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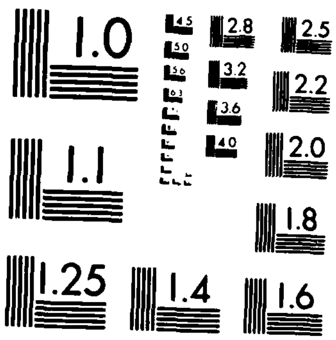
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Operating Characteristics of Log-Normalizer for Weibull and Log-Normal Inputs

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Preface

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The false alarm and detection probabilities of a log-normalizer, subject to either log-normal or Weibull input statistics, are derived for general input signal and noise strengths and number of normalizer samples, N. Plots of the exceedance distribution function versus the threshold, as well as the receiver operating characteristics (i. e., detection probability P_D vs. false alarm probability P_F) are plotted for $N = \infty$, 64, 32, 16 and for various values of the normalizer input deflection statistic d. In addition, simulation results, based on 8.4 million trials, are superposed for purposes of confirming or rejecting the theoretical results.

Plots of the exceedance distribution function are carried out on the extremes of the distribution, to the point where the tail probabilities are

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19. ABSTRACT (Cont'd.)

1E-6. The receiver operating characteristics vary over the range of (P_F, P_D) equal to (1E-6, 1E-6) through (.5, .99). It is found that the theoretical analysis for the log normal input is exact for all N, whereas the approximate theoretical analysis for the Weibull input is sufficiently accurate only for large N, and not on the tails of the distribution.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

N	Number of normalizer samples, (2)
d	Normalizer input deflection, (29),(30)
P_D	Detection probability, (6)
P_F	False alarm probability, (6)
EDF	Exceedance Distribution Function
ROC	Receiver Operating Characteristic
x_n	Input to Log-Normalizer, figure 1
y_n	Normalizer input, figure 1, (1)
z	Normalizer output, figure 1, (4),(28)
y_o	Candidate signal-bearing sample, (4)
$\hat{\mu}(y)$	Sample mean of normalizer input, (2)
$\hat{\sigma}(y)$	Sample standard deviation of normalizer input, (3)
T	Threshold at normalizer output, (5)
w_n	Fundamental random variables, (7)
a,b	Arbitrary scaling and power law, (7)
p_w	Probability density function of random variable w, (8)
Q_w	Exceedance distribution function of random variable w, (9)
Φ	Cumulative distribution function of normalized Gaussian random variable, (12)
ϕ	Probability density function of normalized Gaussian random variable, (12)
$\tilde{v}_n$	Log-distorted version of w_n, (16)
$\mu(\tilde{v})$	Mean of $\tilde{v}$, (17)
$\sigma(\tilde{v})$	Standard deviation of $\tilde{v}$, (17)
v_n	Normalized random variable, (17)
α,β	Constants, (19),(20)

LIST OF SYMBOLS (Cont'd)

$\hat{\mu}(v)$	Sample mean of normalized random variable, (24)
$\hat{\sigma}(v)$	Sample standard deviation of normalized random variable, (24)
v_o	Normalized signal-bearing random variable, (25)
η, υ	Constants, (27)
r	Normalizer input scaling parameter, (29),(30)
H_o	Hypothesis that signal is absent in y_o, (31)
m	Auxiliary constant = $(N-1)/2$, (36)
σ_N^2	Auxiliary variance = $r^2 + 1/N$, (37)
d_r'	Modified deflection parameter, (38),(39)
T_r'	Modified threshold, (38),(39)
$\tilde{\Phi}$	Inverse Φ function, (44)
t	Sample random variable, (46)
γ	Euler's constant = .57721, (49)
$\mu(t)$	Mean of random variable t, (50),(53)
$\sigma(t)$	Standard deviation of random variable t, (50),(54)
h_1, h_2	Auxiliary constants, (52)
ρ	Normalized correlation coefficient, (54),(55)
$\{x_n\}_1^N$	Sequence $x_1, x_2, \ldots, x_N$

OPERATING CHARACTERISTICS OF LOG-NORMALIZER FOR WEIBULL AND LOG-NORMAL INPUTS

INTRODUCTION

The detection of the presence of a weak signal of unknown location and strength in background noise of unknown strength is often accomplished by comparing a candidate signal-bearing detection sample of the observed process with a local estimate of the background level based on N samples of the (hopefully) noise-only process. The local neighborhood can be time, space, or frequency, depending on the application. In order to obtain a stable estimate of the background level, the number of samples, N, should be large; however, if the background is nonstationary, nonhomogeneous, or nonwhite, or if decision and processing time is at a premium, N should be kept as small as reasonably possible. The tradeoff between these conflicting requirements and the dependence on the number of normalizer samples, N, is of interest in this study. Related work is available in [1,2,3].

Since the performance of the normalizer procedure outlined above is adversely affected by the presence of any outliers or noise bursts anywhere in the total of N+1 samples used to make a decision about signal presence or absence in the candidate sample, some form of limiting device

should precede the normalization. The particular combination that we consider in detail here is depicted in figure 1, where ℓn is the natural

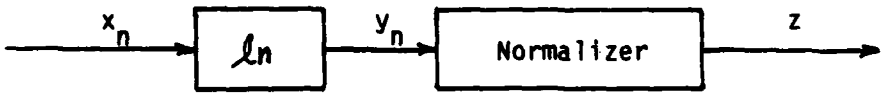

Figure 1. Log-Normalizer

logarithm. The logarithmic device tends to saturate at large input amplitudes and suppress their effect on the normalizer output *z*. *The input* sequence of random variables, $\{x_n\}$, (the detector output sequence), is presumed to be statistically independent and limited to positive values, giving logarithmic output

$$y_n = \ell n(x_n) \ . \tag{1}$$

The particular normalizer we consider here is described as follows: call y_o the candidate signal-bearing sample at the normalizer input, and let $y_1, y_2, \ldots, y_N$ be the N noise-only basis samples employed to extract an estimate of the background level at the normalizer input. Despite the notation, these N samples can (and probably will) surround the candidate sample y_o in location, whether that be time, space, frequency, etc. The sample mean of the normalizer input noise-only samples is

$$\hat{\mu}(y) = \frac{1}{N}\sum_{n=1}^{N} y_n \ , \tag{2}$$

while the corresponding sample standard deviation is defined as

$$\hat{\sigma}(y) = \left\{ \frac{1}{N-1} \sum_{n=1}^{N} [y_n - \hat{\mu}(y)]^2 \right\}^{1/2} . \tag{3}$$

The output of the normalizer in figure 1, that we consider here, is a deflection measure, namely

$$z = \frac{y_0 - \hat{\mu}(y)}{\hat{\sigma}(y)} . \tag{4}$$

The numerator of (4) is an estimate of the difference in means (at the normalizer input) of the candidate signal-plus-noise sample, relative to the noise-only samples; the denominator of (4) is a measure of the inherent fluctuation of the background noise. The dimensionless ratio in (4) eliminates the dependence on absolute levels in favor of relative levels.

The normalizer output z is compared with a threshold T, and a decision made about signal presence in sample y_0 according to the rule

$$\left. \begin{array}{ll} z > T: & \text{declare signal present in } y_0 \\ z < T: & \text{declare signal absent in } y_0 \end{array} \right\} . \tag{5}$$

It is desired to evaluate the false alarm probability and the detection probability, that is,

$$P_F = \text{Prob}\ (z > T \mid \text{signal absent in } y_o),$$
$$P_D = \text{Prob}\ (z > T \mid \text{signal present in } y_o). \tag{6}$$

Both of these probabilities in (6) are exceedance distribution functions, that is, probabilities that random variable z is greater than a threshold value T. We will be interested in plots of (6) versus T, for various signal-to-noise ratios, as well as in plots of P_D versus P_F, the latter known as the receiver operating characteristics.

The normalizer input random variables $\{y_n\}_1^N$ are statistically independent and identically distributed, since inputs $\{x_n\}_1^N$ have been presumed to have these properties. When signal is absent in candidate sample y_o, its probability density function will be taken identical to that of $\{y_n\}_1^N$; however, when signal is present in y_o, its probability density function can be arbitrary.

CLASSES OF INPUT VARIABLES

The noise-only input samples $\{x_n\}_1^N$ to the log-normalizer in figure 1 will be taken from the class of random variables that can be generated from fundamental independent identically-distributed random variables $\{w_n\}$ according to the rule

$$x_n = a\, w_n^b \text{ for } 1 \le n \le N; \quad a > 0,\ b > 0,\ w_n > 0 . \tag{7}$$

The probability density function p_w of $\{w_n\}$ is arbitrary; the total class of random variables defined by (7) is that yielded by allowing parameters a and b to be any positive constants (independent of n).

To fix this concept of a class of random variables, consider the case where w_n is a random variable with the fundamental exponential probability density function

$$p_w(u) = \exp(-u) \quad \text{for } u > 0 . \tag{8}$$

Then the exceedance distribution function of w_n is

$$Q_w(u) = \text{Prob}(w > u) = \int_u^\infty dv\, p_w(v) = \exp(-u) \quad \text{for } u > 0 . \tag{9}$$

It then follows from (7) that the exceedance distribution function of x_n is

$$Q_x(u) = \text{Prob}(x > u) = \text{Prob}(a\, w^b > u) =$$

$$= \text{Prob}\left(w > (u/a)^{1/b}\right) = Q_w\left(\left(\frac{u}{a}\right)^{1/b}\right) = \exp\left[-\left(\frac{u}{a}\right)^{1/b}\right] \quad \text{for } u > 0 \,. \tag{10}$$

But this is just the exceedance distribution function of a Weibull variate* with shape factor 1/b and scaling $(1/a)^{1/b}$. Thus, the class of random variables that can be generated via (7) with arbitrary a,b, from the fundamental exponential probability density function in (8), is the general class of Weibull variates, as given by (10). (If b = 1/2, x is a Rayleigh variate, for example.)

As a second case, let w_n be a random variable with the fundamental log-normal exceedance distribution function

$$Q_w(u) = \Phi(-\ln u) \quad \text{for } u > 0 \,, \tag{11}$$

where

$$\Phi(t) \equiv \int_{-\infty}^{t} ds\, (2\pi)^{-1/2} \exp(-s^2/2) \equiv \int_{-\infty}^{t} ds\, \phi(s) \tag{12}$$

is the cumulative distribution function of a normalized Gaussian random variable. Then by an analogous procedure to (10), the exceedance distribution function of the random variable x_n generated according to (7) is

*See appendix A.

$$Q_x(u) = Q_w\left(\left(\frac{u}{a}\right)^{1/b}\right) = \Phi\left(\frac{\ln(a) - \ln(u)}{b}\right) \quad \text{for } u > 0 \;, \tag{13}$$

which is the exceedance distribution function of a general log-normal variate with additive factor $\ln(a)$ and scaling $1/b$. The probability density function corresponding to (13) is

$$p_x(u) = \frac{1}{bu}\,\phi\left(\frac{\ln(a) - \ln(u)}{b}\right) \quad \text{for } u > 0 \;, \tag{14}$$

where ϕ was defined in (12). Thus, the class of random variables that can be generated via (7) with arbitrary a,b, from the fundamental log-normal exceedance distribution function in (11), is the general class of log-normal variates, as given by (13) and (14).

Returning to the general case for fundamental random variable w_n now, the output of the logarithmic device, (1), is given, upon use of (7), as

$$y_n = \ln(x_n) = \ln a + b \ln w_n = \ln a + b\,\tilde{v}_n \quad \text{for } 1 \le n \le N \;, \tag{15}$$

where we define

$$\tilde{v}_n = \ln w_n \;. \tag{16}$$

Let the mean and standard deviation of $\tilde{v}_n$ be denoted by $\mu(\tilde{v})$ and $\sigma(\tilde{v})$, respectively. Then form the normalized random variable

$$v_n = \frac{\tilde{v}_n - \mu(\tilde{v})}{\sigma(\tilde{v})} \quad \text{for } 1 \le n \le N , \tag{17}$$

which has mean 0 and standard deviation 1. Substitution of (17) in (15) then yields log output

$$y_n = \alpha + \beta \, v_n \quad \text{for } 1 \le n \le N , \tag{18}$$

where constants

$$\alpha = \ln a + b \, \mu(\tilde{v}) \ , \quad \beta = b \, \sigma(\tilde{v}) \ . \tag{19}$$

A direct useful interpretation of these two constants in (19) follows directly from (18); namely, since $\{v_n\}$ are normalized random variables,

$$\alpha = \mu(y), \quad \beta = \sigma(y) \ . \tag{20}$$

These are fundamental statistics of the input to the normalizer in figure 1.

Equations (18) and (19) demonstrate that the output of the logarithmic device in figure 1, for general parameters a,b and random variables $\{w_n\}_1^N$ in transformation (7), can be handled through the linear transformation (18) of a normalized random variable, v_n, with zero mean and unit standard deviation. The new general parameters α,β are given by (19) or (20), where the required statistics are mean

$$\bar{\tilde{v}} = \mu(\tilde{v}) = \mu(\ln w) = \int du \ln(u)\, p_w(u) \ , \tag{21}$$

and mean square

$$\overline{\tilde{v}^2} = \overline{(\ln w)^2} = \int du\, (\ln u)^2\, p_w(u) \ , \tag{22}$$

in terms of the probability density function of input variable w_n in figure 1. Also, except for the specified zero mean and unit standard deviation of v_n in (18), the statistics of v_n are completely arbitrary. Thus, we can use form (18) for the general normalizer input in the following, where α and β are arbitrary constants.

When we now employ (18), the sample quantities in (2) and (3) become

$$\begin{aligned} \hat{\mu}(y) &= \alpha + \beta\, \hat{\mu}(v) \ , \\ \hat{\sigma}(y) &= \beta\, \hat{\sigma}(v) \ , \end{aligned} \tag{23}$$

where

$$\hat{\mu}(v) = \frac{1}{N} \sum_{n=1}^{N} v_n \ ,$$

$$\hat{\sigma}(v) = \left\{ \frac{1}{N-1} \sum_{n=1}^{N} \left[v_n - \hat{\mu}(v) \right]^2 \right\}^{1/2} \ , \tag{24}$$

in terms of the normalized random variables $\{v_n\}_1^N$.

As noted in the paragraph following (6), the probability density function of random variable y_0 is arbitrary for the signal-present hypothesis. Without loss of generality, let

$$y_0 = \eta + \upsilon\, v_0 , \qquad (25)$$

where normalized random variable v_0 has

$$\mu(v_0) = 0, \quad \sigma(v_0) = 1 . \qquad (26)$$

The constants η and υ absorb the absolute scale of y_0; in fact (in analogy with (20)),

$$\eta = \mu(y_0), \quad \upsilon = \sigma(y_0) . \qquad (27)$$

When we now combine (23) and (25) in the normalizer output z, as given by (4), there follows

$$z = \frac{d + r v_0 - \hat{\mu}(v)}{\hat{\sigma}(v)} , \qquad (28)$$

where constants

$$d = \frac{\eta - \alpha}{\beta} , \qquad r = \frac{\upsilon}{\beta} . \qquad (29)$$

Thus, the general output z of the log-normalizer in figure 1, for the general class of inputs (7), can be expressed in the form (28) involving two fundamental constants d, r in (29); an arbitrary normalized random variable v_0; and the sample mean and standard deviation of the normalized random variables $\{v_n\}_1^N$ according to (24).

A useful physical interpretation of the constants in (29) is afforded by utilizing (20) and (27), namely

$$d = \frac{\mu(y_0) - \mu(y)}{\sigma(y)} , \qquad r = \frac{\sigma(y_0)}{\sigma(y)} . \tag{30}$$

Thus, parameter d measures the deflection criterion at the normalizer input, relative to the standard deviation for signal absent. The parameter r is a scaling quantity reflecting the relative fluctuating strengths at the normalizer input. The fundamental analysis problem is now to evaluate the false alarm and detection probabilities specified by (6), for the output random variable given by (28), where d and r are arbitrary constants, and v_0 and $\{v_n\}_1^N$ are normalized random variables.

CONSTANT FALSE ALARM RATE PROPERTY

The general output of the log-normalizer is given by (28). However, for hypothesis H_0 where signal is absent in candidate signal-bearing sample y_0, the statistics of normalizer input y_0 are identical to those of $\{y_n\}_1^N$, as noted in the paragraph under (6). In this case, (30) obviously reduces to

$$d = 0, \quad r = 1 \qquad \text{under } H_0 \; , \tag{31}$$

and (28) yields

$$z = \frac{v_0 - \hat{\mu}(v)}{\hat{\sigma}(v)} \qquad \text{under } H_0 \; , \tag{32}$$

in terms of the independent identically-distributed normalized random variables v_0 and $\{v_n\}_1^N$.

Since v_n in (17) is the normalized random variable corresponding to logarithmic distortion (16) of fundamental random variable w_n, and does not involve a or b, all scale factors involving constants a and b in (7) have disappeared in output z in (32), under hypothesis H_0. This means that the false alarm probability P_F in (6) cannot depend on a,b; put another way, the false alarm probability for the log-normalizer of figure 1, subjected to

the class of inputs given by (7), is the <u>same</u> for all members of the class, regardless of the values of a and b. Since the sample mean and sample standard deviation in (32) still depend on N, as seen by reference to (24), the false alarm probability will necessarily be a function of N, as well as depend on the *particular* probability density function of independent identically-distributed normalized random variables v_0 and $\{v_n\}$. However, in general, there will be no need to investigate the false alarm probability for the general Weibull class in (10), but instead we can confine attention to the fundamental exponential probability density function of w_n as given by (8). Of course, v_n must then be the normalized random variable, as given by (16), (17), (21), and (22). More details on the statistics of Weibull variates and their logarithmically-distorted counterparts are given in appendix A.

A similar statement can be made with regard to the fundamental log-normal exceedance distribution function given by (11). In fact, the logarithmically transformed input, (16), to the normalizer has exceedance distribution function

$$Q_{\tilde{v}}(u) = \text{Prob}(\tilde{v} > u) = \text{Prob}(\ln w > u) = \text{Prob}(w > \exp(u)) =$$

$$= Q_w(\exp(u)) = \Phi(-u) \quad \text{for all } u \,. \tag{33}$$

But this is the exceedance distribution function of a zero-mean unit-variance Gaussian random variable. Thus, $\tilde{v}$ of (16) is already a normalized random variable, and Gaussian at that. Therefore, decision

variable z in (32) involves a collection of N+1 independent identically-distributed zero-mean unit-variance random variables v_0 and $\{v_n\}_1^N$. Again, the false alarm probability can only depend on N, and not on scale factors a and b in (13) and (14). Of course, the detection probability (6), as applied to (28), will depend additionally on parameters d and r in (29) and (30).

In summary, the log-normalizer in figure 1 will possess constant false alarm rate properties, that is, the same false alarm probability for all the members of the class of random variables generated according to (7), regardless of the values of a and b.

PERFORMANCE FOR LOG-NORMAL INPUT x

The problem of interest in this section is the evaluation of detection probability (6) for the decision variable z given by (28), when normalized random variables $\{v_n\}_1^N$ are independent identically-distributed zero-mean unit-variance Gaussian random variables; this is the case discussed in (33) et seq. Although the probability density function of normalized random variable v_0 is arbitrary, we will also take it here to be zero-mean unit-variance Gaussian. Reference to (15), (17), and (25) reveals that this is tantamount to assuming that the normalizer input $\{y_n\}_1^N$ in figure 1 is Gaussian with arbitrary mean and variance, while y_0 is also Gaussian with different arbitrary mean and variance. All these arbitrary parameters are collected together in (28) in the parameters d and r, according to (30). This situation is also equivalent to assuming log-normal excitations at the input of the log-normalizer of figure 1.

From (6) and (28), since $\hat{\sigma} > 0$,

$$P_D = \mathrm{Prob}(z > T) = \mathrm{Prob}\left(d + rv_0 - \hat{\mu}(v) > T\,\hat{\sigma}(v)\right) , \tag{34}$$

where $\hat{\mu}(v)$ and $\hat{\sigma}(v)$ are given by (24). It is shown in appendix B that $\hat{\mu}(v)$ and $\hat{\sigma}(v)$ are statistically independent, with probability density functions given by (B-16) and (B-20), respectively, (setting $\mu = 0$, $\sigma = 1$) as

$$p_{\mu}(u) = (2\pi/N)^{-1/2} \exp(-\frac{N}{2} u^2) \text{ for all } u \tag{35}$$

and

$$p_{\hat{\sigma}}(u) = \frac{2\, m^m\, u^{2m-1} \exp(-mu^2)}{\Gamma(m)} \quad \text{for } u > 0; \quad m = \frac{N-1}{2} . \tag{36}$$

Now the quantity $d + rv_0 - \hat{\mu}(v)$ in (34) is a Gaussian random variable with mean d and variance $r^2 + 1/N \equiv \sigma_N^2$; see (B-16) or (35). Considering $\hat{\sigma}$ fixed for the moment, the conditional detection probability in (34) is then

$$P_{DC} = \int_{T\hat{\sigma}}^{\infty} dt \left(2\pi\sigma_N^2\right)^{-1/2} \exp\left[-\frac{(t-d)^2}{2\sigma_N^2}\right] = \Phi\left(\frac{d - T\hat{\sigma}}{\sigma_N}\right) , \tag{37}$$

upon use of (12). Averaging this result over the probability density function (36) of $\hat{\sigma}$, we have the unconditional detection probability

$$P_D = \int_0^{\infty} du\, \Phi\left(\frac{d - Tu}{\sigma_N}\right) \frac{2\, m^m\, u^{2m-1} \exp(-mu^2)}{\Gamma(m)} =$$

$$= \int_0^{\infty} dw\, \frac{w^{N-2} \exp(-w^2/2)}{2^{\frac{N-3}{2}}\, \Gamma\left(\frac{N-1}{2}\right)}\, \Phi(d_r' - T_r' w) , \tag{38}$$

where $N \geq 2$ and

$$d_r' \equiv \frac{d}{\sigma_N} = \frac{d}{\sqrt{r^2 + 1/N}},$$

$$T_r' \equiv \frac{T}{\sqrt{N-1}\,\sigma_N} = \frac{T}{\sqrt{N-1}\sqrt{r^2 + 1/N}}. \tag{39}$$

The fundamental parameters upon which P_D depends are

N, number of normalizer samples;

T, threshold at normalizer output;

d, deflection criterion (30);

r, scaling (30) . (40)

However, they show up, in integral result (38), collapsed into the three variables N, d'_r, T'_r.

For signal absent, we have $d=0$ and $r=1$, as noted in (31). Then (38) and (39) reduce to the false alarm probability

$$P_F = \int_0^\infty dw \; \frac{w^{N-2} \exp(-w^2/2)}{2^{\frac{N-3}{2}} \Gamma\left(\frac{N-1}{2}\right)} \; \Phi\left(-T\sqrt{\frac{N}{N^2-1}}\,w\right), \tag{41}$$

which depends only on N and T. Thus, given a particular number N of normalizer samples, threshold T can be selected to realize a specified value of false alarm probability P_F. This applies for the complete class of log-normal inputs, (13) or (14), into the log-normalizer in figure 1, and can be achieved without knowledge of a or b.

As $N \to \infty$, we have

$$\hat{\sigma} \to 1, \; \sigma_N \to r \quad \text{as } N \to \infty, \tag{42}$$

giving from (37),

$$P_D = \Phi\left(\frac{d - T}{r}\right), \qquad P_F = \Phi(-T) \qquad \text{for } N = \infty. \qquad (43)$$

This yields

$$\tilde{\Phi}(P_D) = \frac{1}{r}\left[d + \tilde{\Phi}(P_F)\right] \qquad \text{for } N = \infty, \qquad (44)$$

where $\tilde{\Phi}$ is the inverse Φ-function; this last result is useful for plotting receiver operating characteristics on normal-probability paper. It illustrates that for $N=\infty$, those curves are straight lines with slope 1/r and offset d/r at P_F=.5.

The actual numerical evaluations of false alarm probability (41) and detection probability (38) are undertaken in appendix C. The inputs to the functions considered there are the 3 parameters N, T_r', d_r' as given by (39), rather than the 4 fundamental parameters N,T,d,r listed in (40). This is no limitation, since for any given values of N,T,d,r, the quantities N, T_r', d_r' can be easily calculated via (39) and used as inputs to the procedures in appendix C.

PERFORMANCE FOR WEIBULL INPUT *

Here we want to evaluate detection probability (6) for the decision variable z given by (28), when normalized random variables $\{v_n\}_1^N$ are independent identically-distributed zero-mean unit-variance log-distorted Weibull variates. Before we do that, we observe that detection probability (6) can be expressed generally as

$$P_D = \text{Prob}(z > T) = Q_z(T) = \text{Prob}\left(\frac{d + rv_o - \hat{\mu}(v)}{\hat{\sigma}(v)} > T\right) =$$

$$= \text{Prob}\left(v_o > \frac{t - d}{r}\right) = \int du\; Q_{v_o}\left(\frac{u - d}{r}\right) p_t(u) \;, \tag{45}$$

where we used (28) and defined random variable

$$t = \hat{\mu}(v) + T\,\hat{\sigma}(v) \tag{46}$$

in terms of the sample quantities in (24). The separation of functions in the last form in (45) is due to the fact that random variables v_o and t are statistically independent of each other. When signal is absent, then d=0, r=1 according to (31), and (45) reduces to false alarm probability

$$P_F = \int du\; Q_{v_o}(u)\; p_t(u) \;. \tag{47}$$

In the special case where $N \to \infty$, that is, a very large number of samples used in the normalizer, the sample quantities in (24) approach the true mean 0 and standard deviation 1 of normalized random variables $\{v_n\}$, and t tends to the constant T. Then (45) and (47) reduce to

$$P_D = Q_{v_0}\left(\frac{T - d}{r}\right), \quad P_F = Q_{v_0}(T) \quad \text{for } N = \infty. \tag{48}$$

This limiting case can be used as a comparison with practical cases where N is large, but not infinite.

We now specialize the above general results to the case of log-distorted Weibull variates $\{v_n\}_1^N$. Although the probability density function of normalized random variable v_0 is arbitrary, we will also take it here to be a normalized log-distorted Weibull variate. In this case, the exceedance distribution function of random variable v_0 is given by (A-17) as

$$Q_v(u) = \exp\left[-\exp\left(-\gamma + \frac{\pi}{\sqrt{6}}u\right)\right] \quad \text{for all } u, \tag{49}$$

where γ=.57721 is Euler's constant. Thus, the detection and false alarm probabilities for $N = \infty$, as given generally by (48), are immediately available upon use of (49).

The probability density function of random variable t defined in (46) is a much more difficult task for finite N. To make any analytic progress, we have had to assume that t is Gaussian; this can be expected to be a fair approximation if the number of normalizer samples N entering the sample

quantities in (24) is large, according to the central limit theorem. However, we can anticipate that the approach of random variable t to normality will be faster near its mean, but considerably slower on the tails. This can lead to a significant bias in the calculation of small false alarm probabilities.

Thus, our assumption is that the probability density function of t in (46) is given by

$$p_t(u) = [2\pi\ \sigma^2(t)]^{-1/2} \exp\left[-\frac{(u-\mu(t))^2}{2\ \sigma^2(t)}\right]. \tag{50}$$

Combining (49) and (50) in (45), the detection probability is given (approximately) by

$$P_D = \int du \exp\left[-\exp\left(-\gamma + \frac{\pi}{\sqrt{6}}\frac{u-d}{r}\right)\right] *$$

$$* \left[2\pi\ \sigma^2(t)\right]^{-1/2} \exp\left[-\frac{(u-\mu(t))^2}{2\ \sigma^2(t)}\right] =$$

$$= (2\pi)^{-1/2} \int dx \exp\left[-\frac{x^2}{2} - \exp\left(h_1 + h_2\ x\right)\right] , \tag{51}$$

where constants

$$h_1 = -\gamma + \frac{\pi}{\sqrt{6}}\frac{\mu(t)-d}{r} , \qquad h_2 = \frac{\pi}{\sqrt{6}}\frac{\sigma(t)}{r} . \tag{52}$$

The false alarm probability follows upon setting d = 0, r = 1 in (52).

The fundamental parameters in integral result (51) are d and r, along with mean $\mu(t)$ and standard deviation $\sigma(t)$ of random variable t. The

complexity of random variable t, defined by (46) and (24), precludes us from evaluating mean $\mu(t)$ and standard deviation $\sigma(t)$ exactly. However, numerous simulations, each consisting of 100,000 trials, enabled us to extract the following rather accurate rules of thumb for the statistics of t.

First of all, for general definition (46), we have mean

$$\mu(t) = \mu\{\hat{\mu}(v)\} + T\,\mu\{\hat{\sigma}(v)\} \,, \tag{53}$$

and variance

$$\sigma^2(t) = \sigma^2\{\hat{\mu}(v)\} + T^2\,\sigma^2\{\hat{\sigma}(v)\} + 2T\,\sigma\{\hat{\mu}(v)\}\,\sigma\{\hat{\sigma}(v)\}\,\rho \,, \tag{54}$$

where ρ is the normalized correlation coefficient between $\hat{\mu}(v)$ and $\hat{\sigma}(v)$.

The simulation results alluded to above, for normalized random variables $\{v_n\}$ in (24) being log-distorted Weibull variates, are given by

$$\mu\{\hat{\mu}(v)\} = 0 \qquad \mu\{\hat{\sigma}(v)\} \cong 1 - \frac{.39}{N}$$

$$\sigma^2\{\hat{\mu}(v)\} = \frac{1}{N} \qquad \sigma^2\{\hat{\sigma}(v)\} \cong \frac{1.05}{N + 1.5}$$

$$\rho \cong -.55 \,. \tag{55}$$

Observe the large value of ρ, in contrast with the earlier case of a Gaussian input to the normalizer, where the sample mean and sample standard deviation were not only uncorrelated but in fact independent; see appendix B.

The quantities in (55) depend solely on N; when used in (53) and (54), it is seen that $\mu(t)$ and $\sigma(t)$ in (54) depend on both N and threshold T. Thus, the totality of fundamental parameters of relevance in detection probability (51) is d,r,N,T, just as for the Gaussian case in (40). Analytic evaluation of integral (51) is impossible; accordingly, numerical integration was employed.

An alternative approximation to the statistics of random variable t of (46) is undertaken in appendix D. Namely, for large threshold values T, where random variable t is dominated by sample standard deviation $\hat{\sigma}(v)$, it might be thought that a χ-approximation would have wider applicability than a Guassian one. This is indeed true, as will be demonstrated by the numerical results to follow; however, for small false alarm probabilities, the χ-approximation also falls short.

GRAPHICAL RESULTS

In this section, we will present graphical results for the exceedance distribution functions and receiver operating characteristics for both the Gaussian and the log-distorted Weibull inputs to the normalizer in figure 1. This corresponds, respectively, to log-normal and Weibull inputs (that is, detector outputs) to the logarithmic device in figure 1. The theoretical results of the previous two sections are augmented by simulation results, each based upon $2^{23} = 8.4$ million trials of decision variable (normalizer output) z given by (28) and (24).

The scaling parameter r, that is, the ratio of standard deviations in (30), is taken at 1 for all these results, in order to keep the number of plots at a reasonable level. The deflection parameter d in (30) is varied from 0 to values large enough to sweep out the important range of detection and false alarm probabilities of interest. The number of normalizer samples, N, is taken at the values ∞, 64, 32, 16, which appears to cover the most important range of practical use. Threshold value T in exceedance probabilities (6) is allowed to vary widely, so that the full range of detection and false alarm probabilities can be observed.

GAUSSIAN INPUT TO NORMALIZER

The exceedance distribution function (EDF), defined in (6), for $N=\infty$ is plotted in figure 2 versus threshold T, for deflection parameter d taking on values

$$d = 0(1)9 = 0,1,2,3,4,5,6,7,8,9 \ . \qquad (56)$$

The arrow on the figure indicates the direction of increasing d; thus d=0, which is the false alarm probability, corresponds to the curve at the lower left. The results in figure 2 are based on (43); since the ordinate is according to a normal probability rule, these curves are perfectly straight lines. Exceedance probabilities ranging from 1E-6 to .999999 are covered when threshold T is varied over the range -5,5.

When N takes on the values 64, 32, 16, the corresponding results are displayed in figures 3, 4, 5, respectively. These graphs were obtained from (38)-(41), implemented by the procedures in appendix C. The exceedance distribution function for N=64 in figure 3 is fairly close to that for $N=\infty$ in figure 2; however, by the time the number of normalizer samples N has decreased to 16 in figure 5, significant curvature has developed in the results.

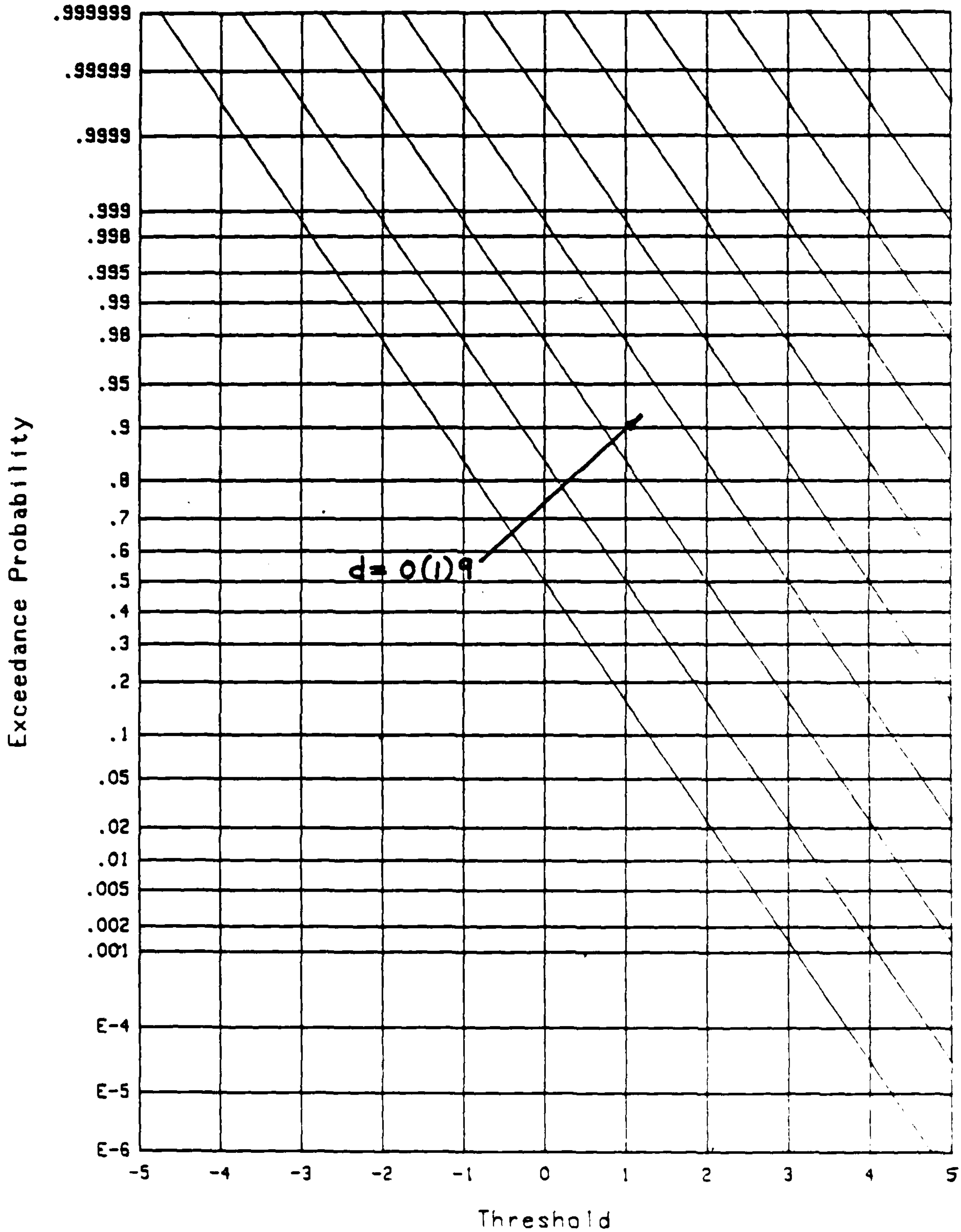

Figure 2. EDF for N = ∞; Gaussian

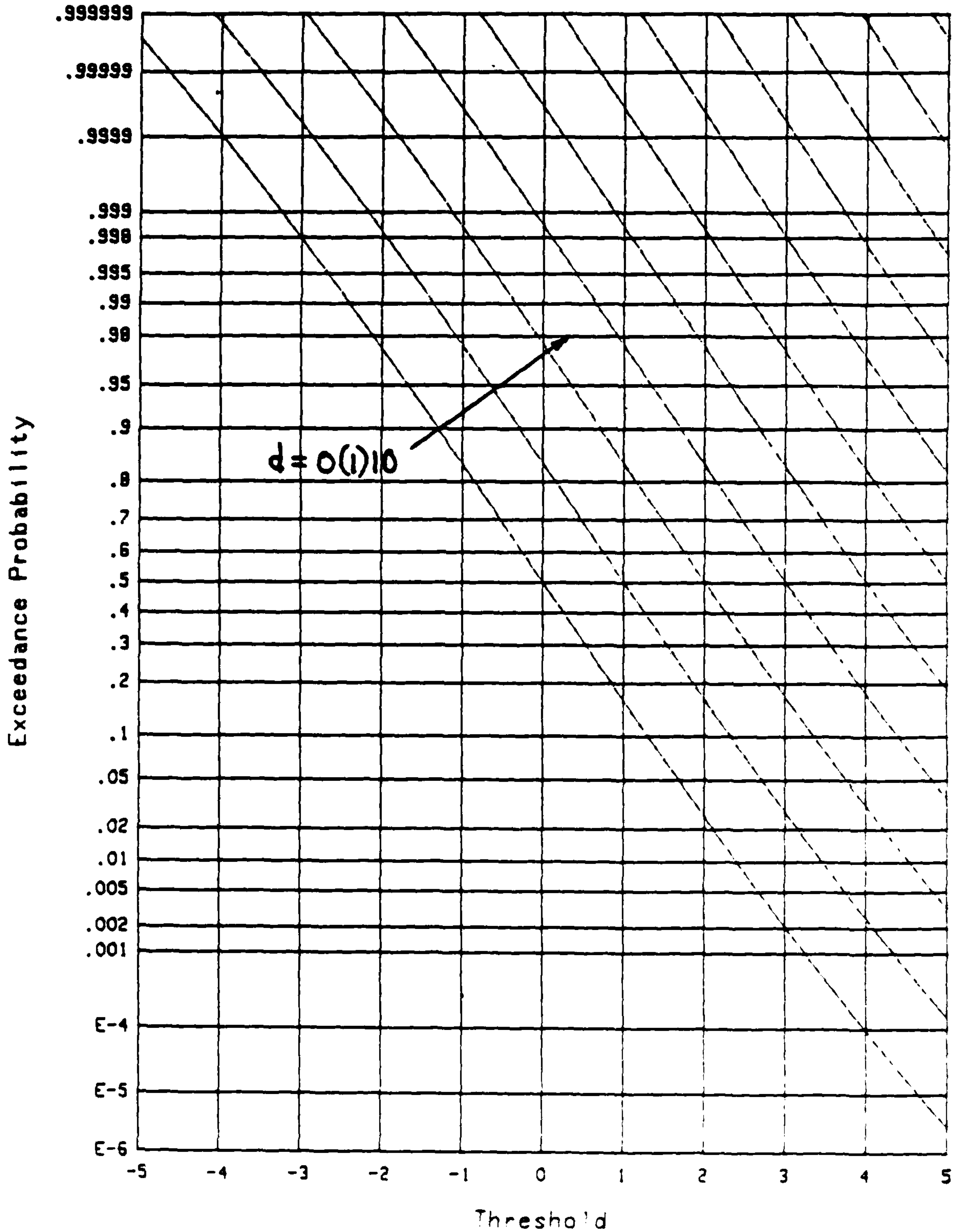

Figure 3. EDF for N = 64; Gaussian

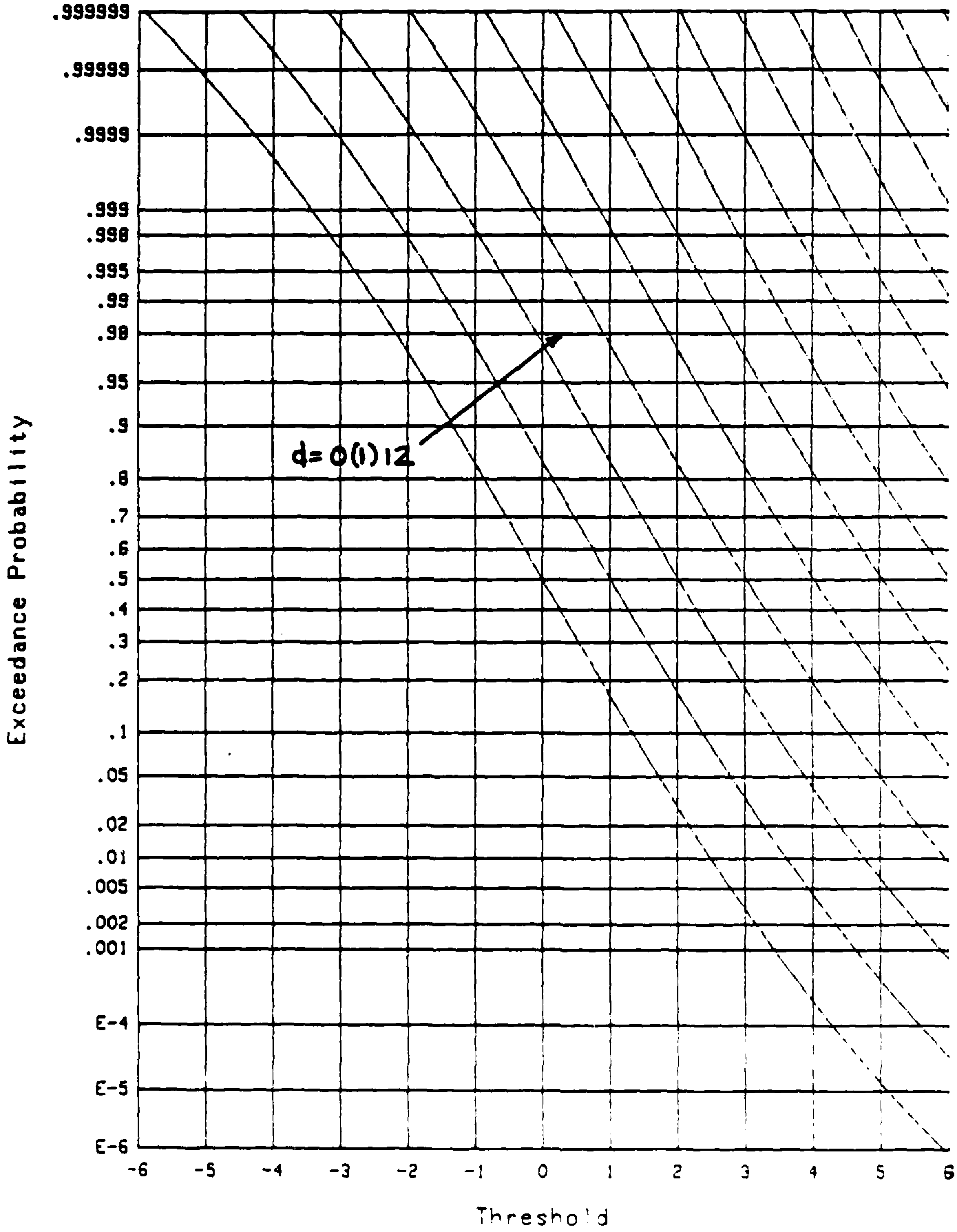

Figure 4. EDF for N = 32; Gaussian

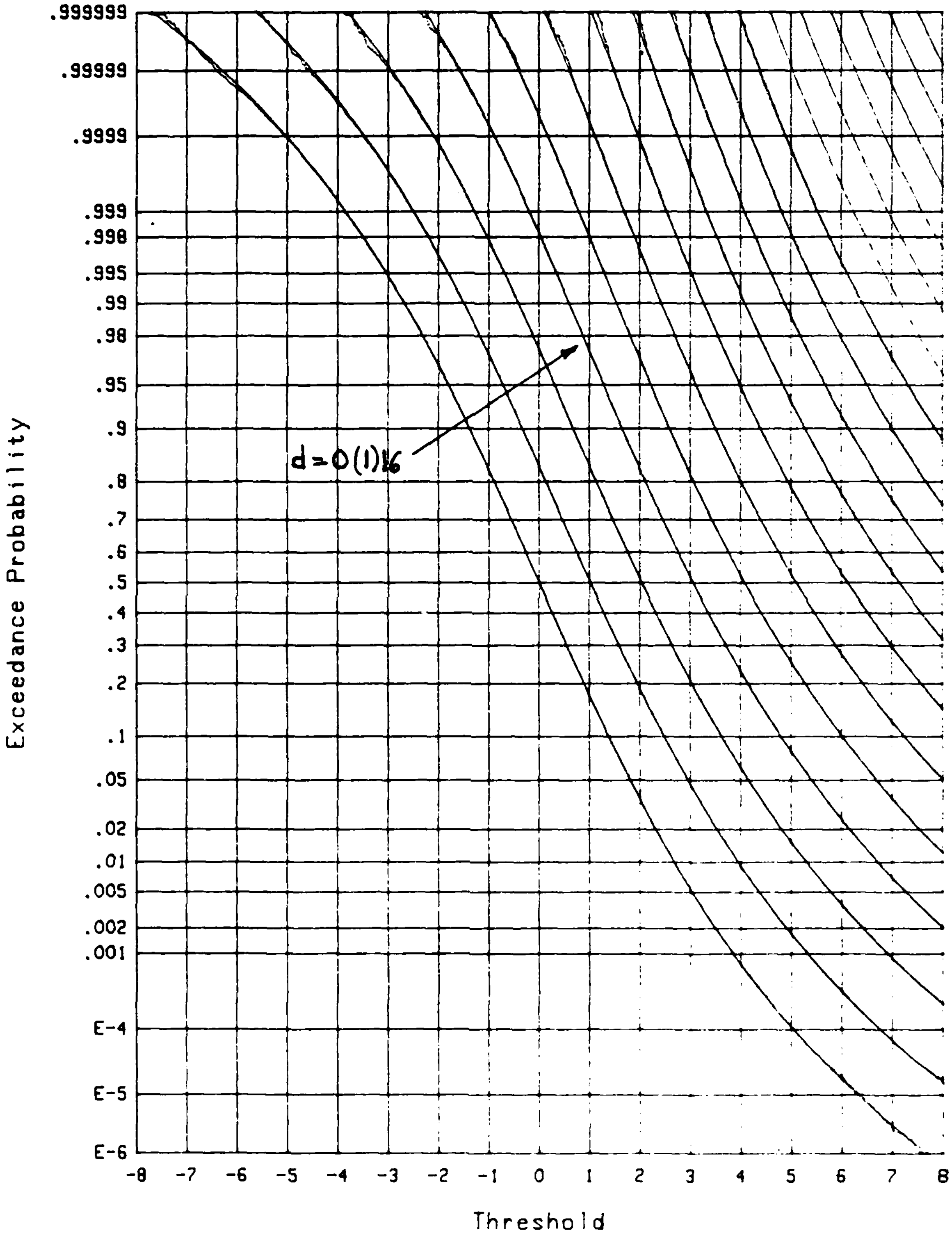

Figure 5. EDF for N = 16; Gaussian

Superposed in figure 5 are 11 simulation results for d=0(1)10, each based upon 8.4 million independent trials. Due to the large number of trials, the theoretical and simulation results are indistinguishable, except near the extremes of probability 1E-6 and .999999, where the jagged character of simulation results is manifested. This close agreement of results not only confirms the theoretical analysis but also lends credence to the use of simulation for the estimation of probabilities out on the tails of the distribution, provided that enough trials are conducted.

Figures 3, 4, 5 furnish information which enables the selection of the required threshold T to realize a specified false alarm probability for N = 64, 32, 16, respectively. For example, figure 5 with d = 0 indicates that to realize a false alarm probability of 1E-5 for N = 16, threshold T in (6) must be chosen as 6.3.

When threshold T is eliminated, and the detection probability plotted versus the false alarm probability, we obtain the receiver operating characteristics (ROC). The result for $N=\infty$ is given in figure 6, where both the abscissa and ordinate are plotted according to a normal probability scale. Deflection parameter d varies over the range

$$d = 0(.5)7.5 = 0,.5,1,1.5,\ \ldots,\ 7,7.5\ . \tag{57}$$

The arrow again points in the direction of increasing d; thus d=0 is the curve on the lower right. These curves are precisely the straight lines indicated by (44) with r=1.

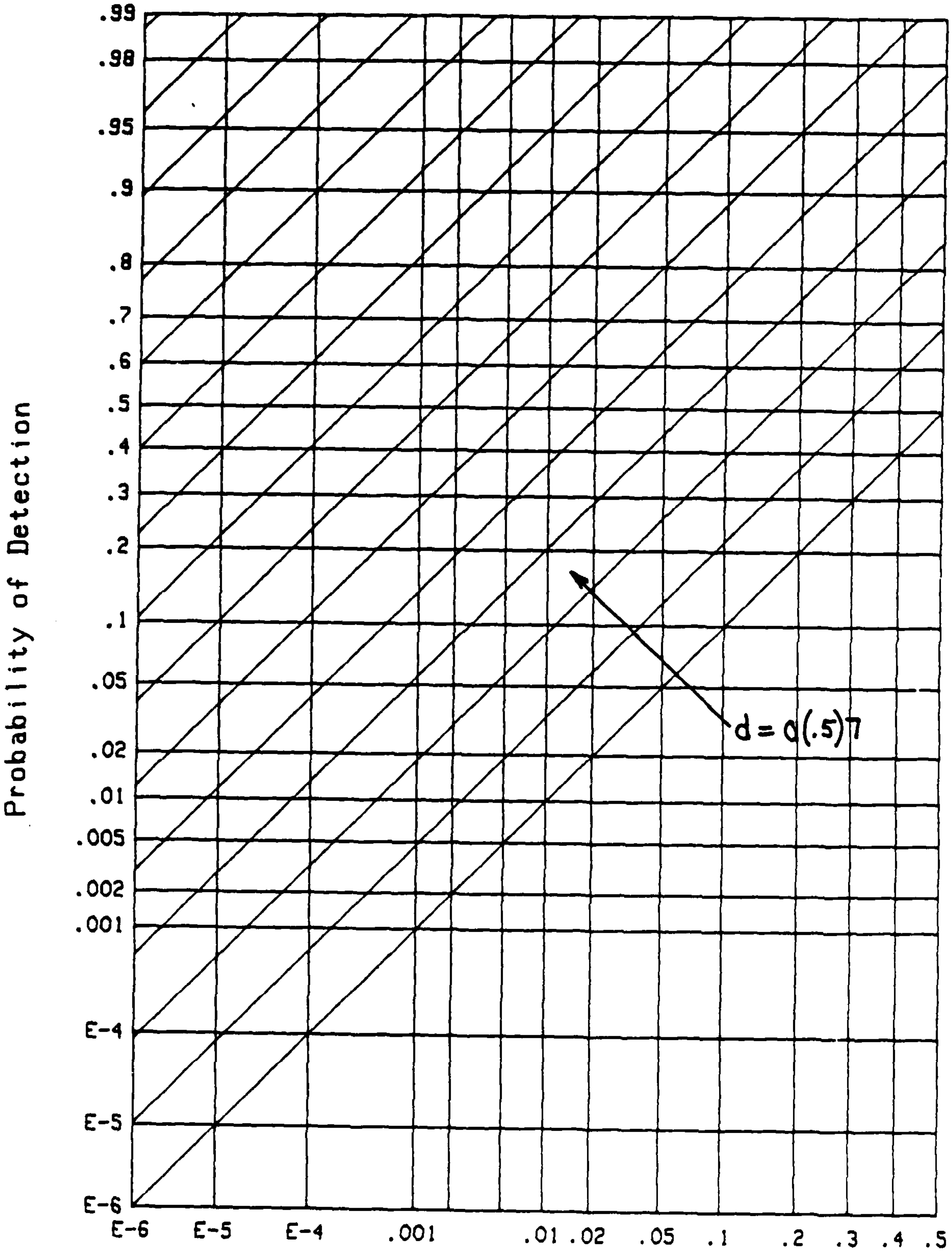

Figure 6. Probability of False Alarm

ROC for N = ∞; Gaussian

Corresponding receiver operating characteristics for $N = 64, 32, 16$ are presented in figures 7, 8, 9. The detection and false alarm probabilities both range down to 1E-6, while the upper limits have been truncated at .99 and .5, respectively. Values beyond these limits can be obtained from the earlier figures 2 through 5.

Superposed in figure 9 are ten simulation results for $d = 1(1)10$. Again, except for the small probability regions like $P_F < 1E-5$, the theoretical and simulation results are indistinguishable and overlay each other. It will be noticed that a characteristic wiggle in the receiver operating characteristics is duplicated for every simulation result, at a constant value of false alarm probability; for example, see the triangular bump in all 10 simulation results at $P_F \approx 1E-6$. The reason for this behavior is that when random variable z in (28) was simulated, the random numbers employed in (24) to generate $\hat{\mu}$ and $\hat{\sigma}$ were not changed when different d values were considered in (28). The reason for this deliberate choice was economy of computer execution time; that is, the time-consuming task of computation of (24) was done once for each trial, and used in (28) for all of the d values of interest. This repeated use of the same $\hat{\mu},\hat{\sigma}$ values for different d values gives a persistent systematic perturbation to the estimated receiver operating characteristics at a fixed false alarm probability. However, for 8.4 million trials, this bias is small, even for the rare events with probabilities greater than 1E-6, and was deemed acceptable in light of the greatly increased computer time required for the alternative approach of regeneration of $\hat{\mu}$ and $\hat{\sigma}$.

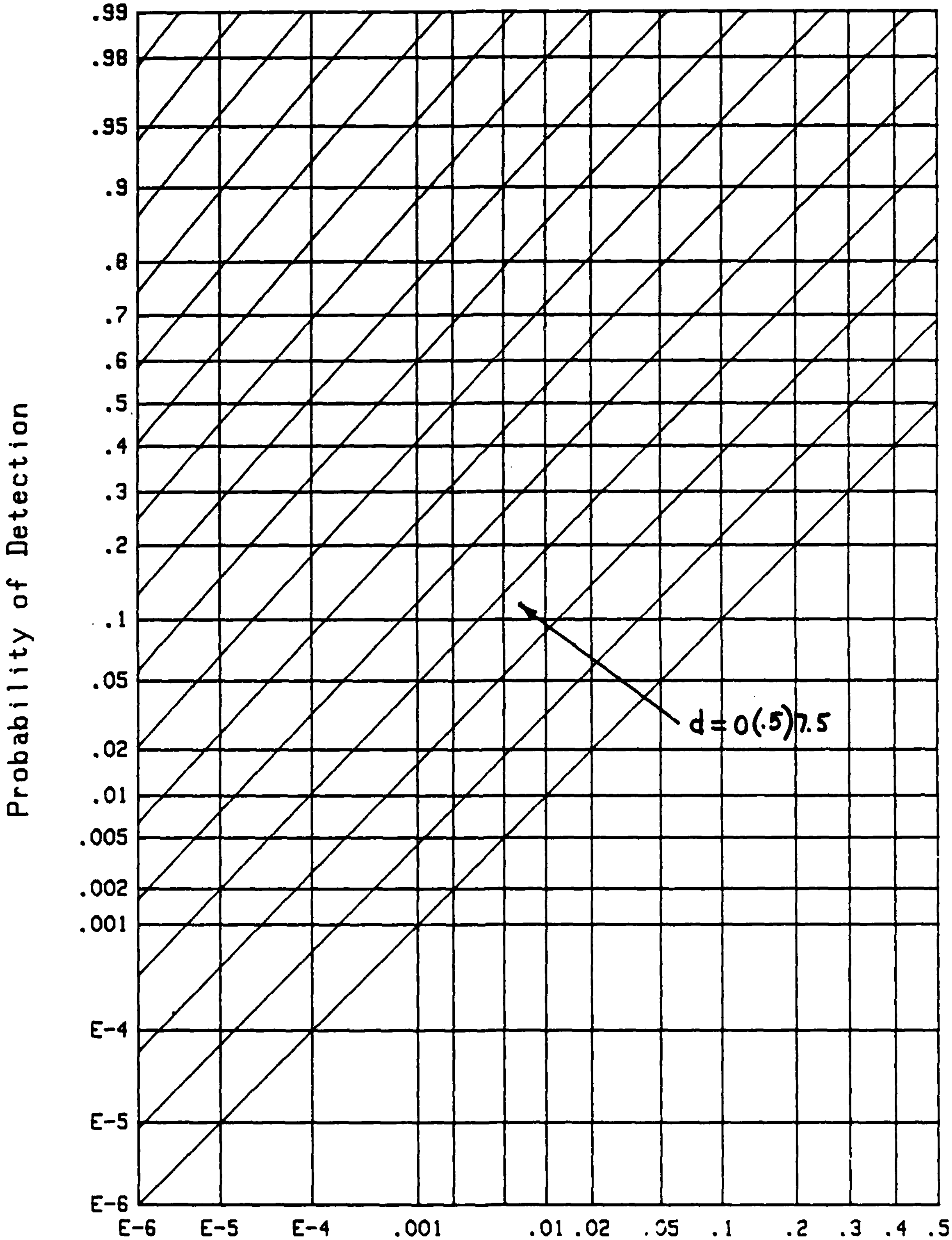

Figure 7. Probability of False Alarm

ROC for N = 64; Gaussian

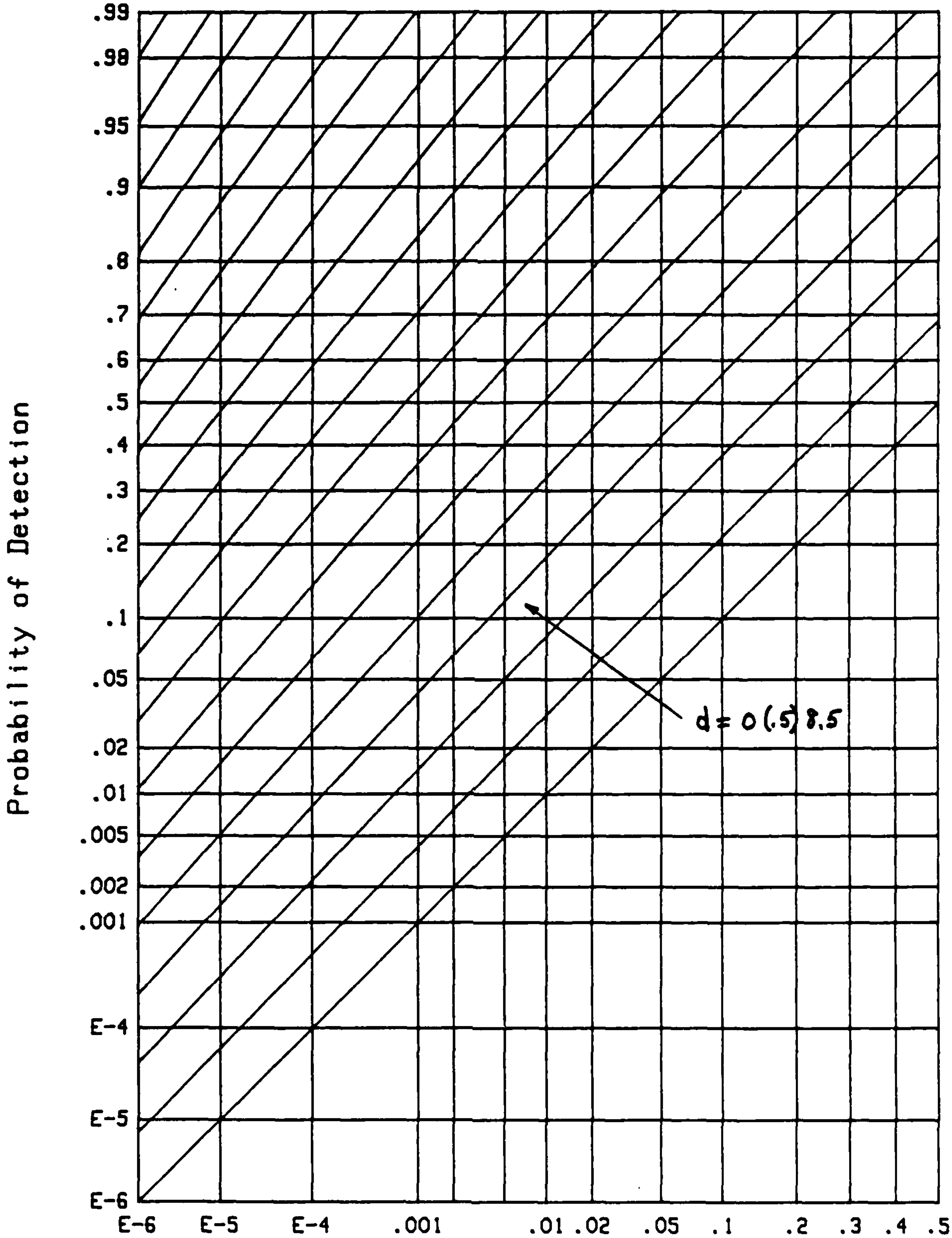

Figure 8. Probability of False Alarm

ROC for N = 32; Gaussian

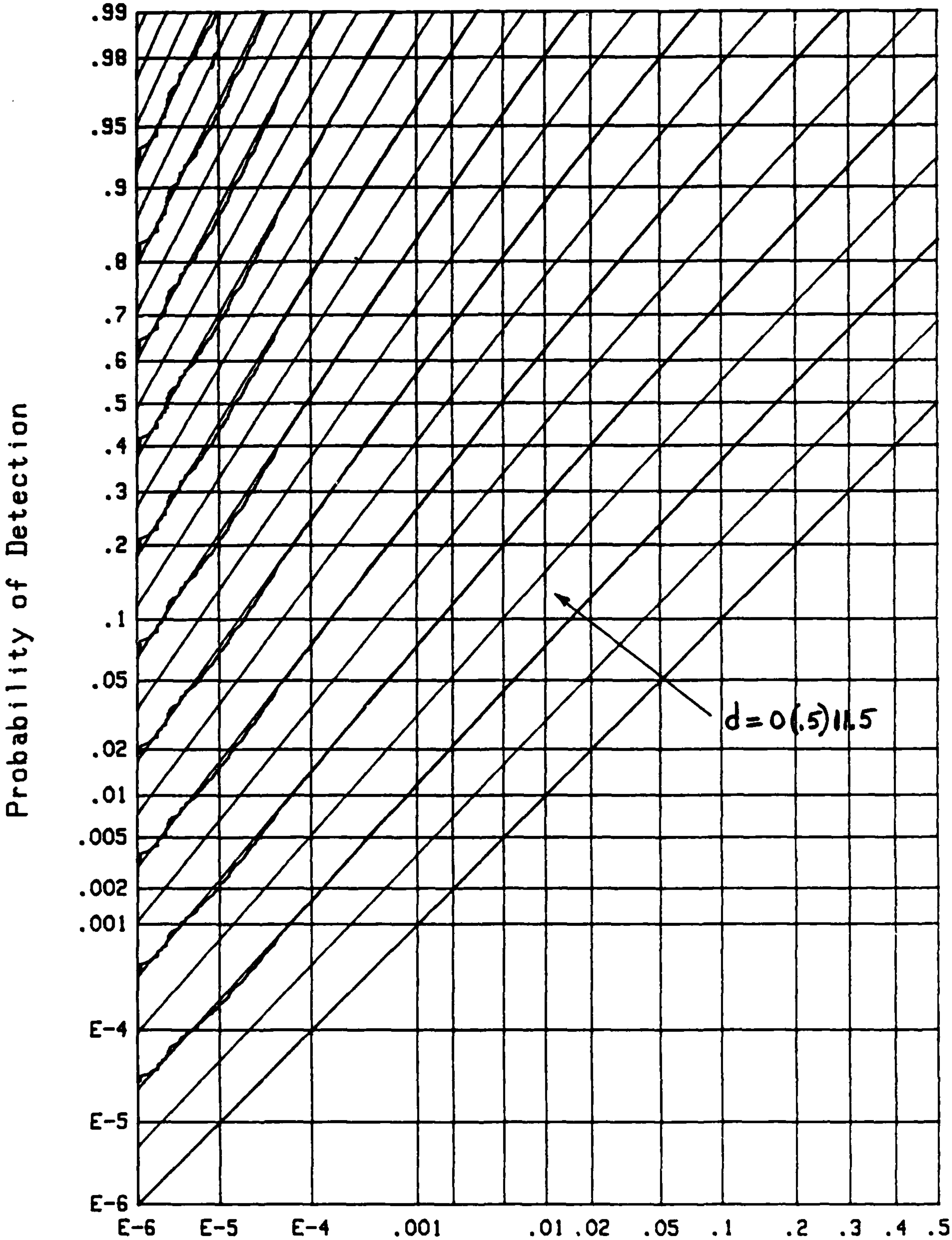

Figure 9. Probability of False Alarm

ROC for N = 16; Gaussian

As an example of the use of figures 6 through 9, the values of deflection d required to realize P_F = 1E-5 and P_D = .5 are

$$d = 4.3,\ 4.7,\ 5.1,\ 6.2 \quad \text{for } N = \infty,\ 64,\ 32,\ 16\ , \tag{58}$$

respectively. The cost of reducing N from ∞ to 16 is that d must be increased by the factor 6.2/4.3 = 1.44; whether this is tolerable depends on the application. The relation of deflection parameter d to any system input signal-to-noise ratio depends on the particular processor form preceding the logarithmic device in figure 1, and must be left to the user and his particular application.

LOG-WEIBULL INPUT TO NORMALIZER

When the input to the normalizer is a log-distorted Weibull variate, the performance is markedly different. The exceedance distribution function for N=∞ is displayed in figure 10 and has a significant curvature when plotted on normal probability paper; these results are based upon the use of (48) and (49). The use of the notation 'Extreme' is explained in (A-7) et seq.

When N is decreased to 64, the corresponding exceedance distribution functions are given in figure 11. Due to the questionable assumptions required in the theoretical analysis of this case and used in (50) et seq., simulation results were also superposed for the values d = 0(.5)5. Agreement in the mid-range of probabilities is excellent. At the low end of the probability range, near 1E-6, the simulation results indicate a systematically lower exceedance probability than predicted by theory.

This erratic trend of the theoretical approximation is continued and accented in figure 12 for N=32 and in figure 13 for N=16. In fact, in the latter case, for threshold T=5, the simulation indicates exceedance probabilities for d=0 that are more than 2 orders of magnitude smaller than the theory predicts; see bottom right of figure 13. The discrepancies at the high end of probabilities are also considerable, as seen at the top left of the figure.

Also added to this particular figure is the result of using the χ-approximation for random variable t of (46), as detailed in appendix D. Although the improvement in probability values is over an order of magnitude, there is still another order of magnitude error left in this alternative approximate approach. The reason for the difficulty in the theoretical analysis is two-fold: (1) values of N like 16 or 32 are not large enough for the central limit theorem to have developed substantial accuracy on the tails; (2) the probability density function of a log-distorted Weibull variate, as given by (A-7), is distinctly non-Gaussian on the tails. The decay of (A-7) on the positive tail is much faster than Gaussian, while that on the negative tail is slower, being only exponential.

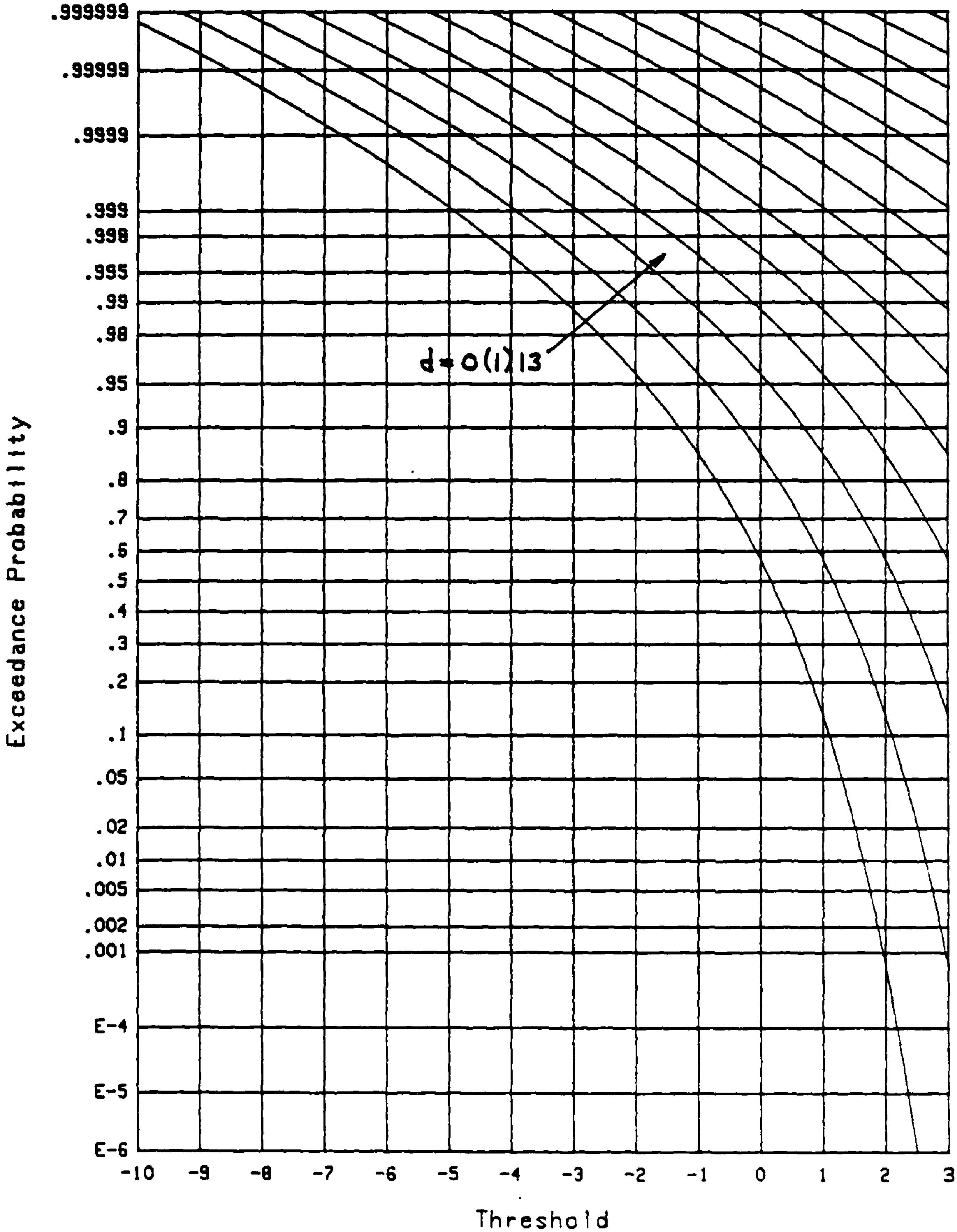

Figure 10. EDF for N = ∞; Extreme

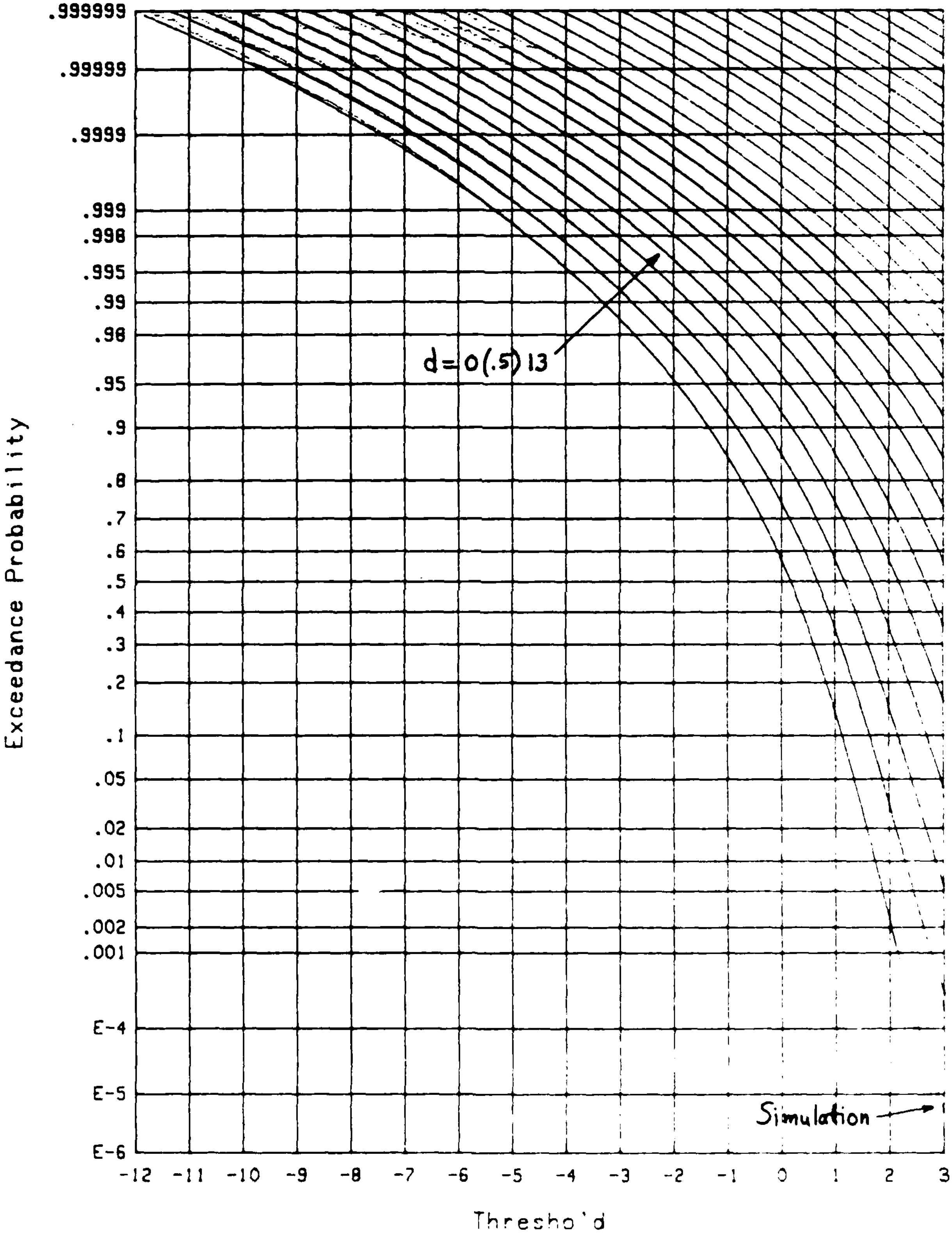

Figure 11. EDF for N = 64; Extreme

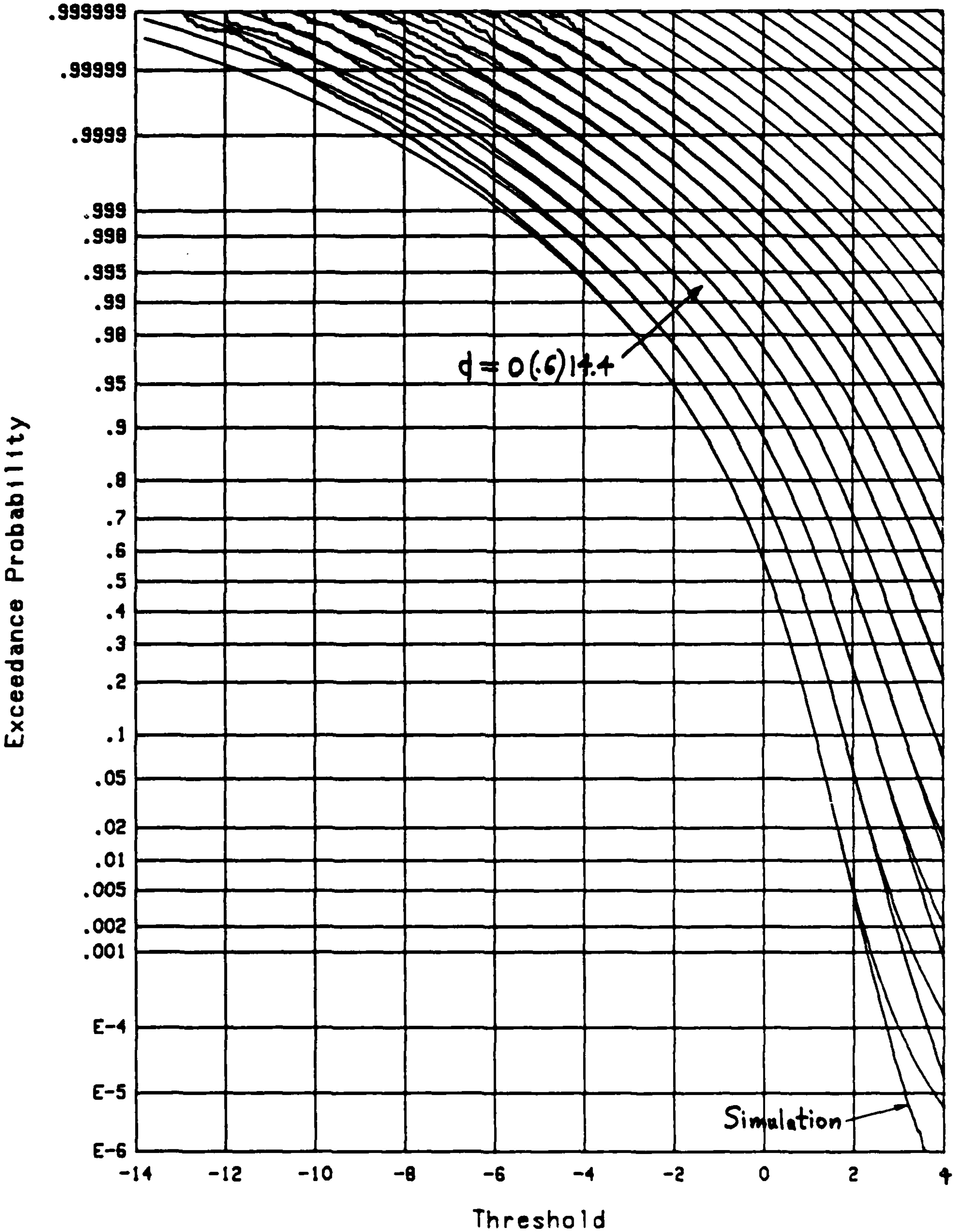

Figure 12. EDF for N = 32; Extreme

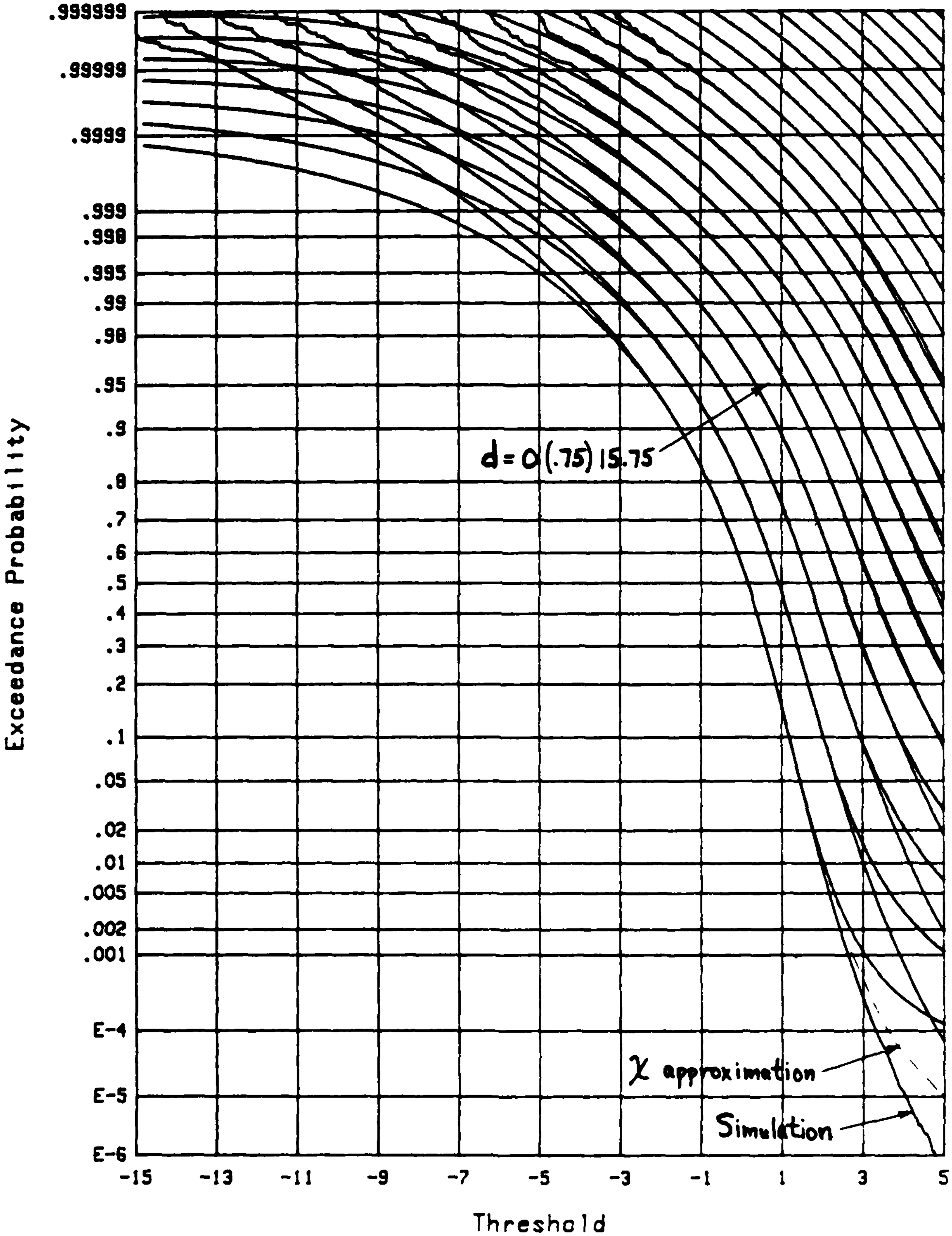

Figure 13. EDF for N = 16; Extreme

The receiver operating characteristics for $N=\infty$ are given in figure 14, while those for N = 64, 32, 16 are given in figures 15 through 17, respectively. The discrepancy between theory and simulation becomes progressively larger as N decreases, reaching the point in figure 17 where the theory is entirely invalid for false alarm probabilities less than approximately .001. The reason for the severe dip of the theoretical curves to the left of each figure is the inadequacy of the false alarm probability approximation, it being much too large for the larger threshold values; see bottom right of figure 13. On the other hand, the simulation results in these figures are all based on 8.4 million independent trials, making them trustworthy well down near the 1E-6 level of probability plotted here.

As an example of the use of figures 14 through 17, the values of d required to realize probabilities P_F = 1E-5 and P_D = .5 are

$$d = 2.2,\ 2.6,\ 2.9,\ 3.8 \quad \text{for } N = \infty,\ 64,\ 32,\ 16\ , \tag{59}$$

respectively. The latter three values are extracted from the simulation results in figures 15 through 17. Direct comparison of the absolute levels in (59) with the corresponding Gaussian results in (58) is not valid, because the shapes of the input probability density functions in the two cases are markedly different and are more important than the deflection criterion, defined by (30).

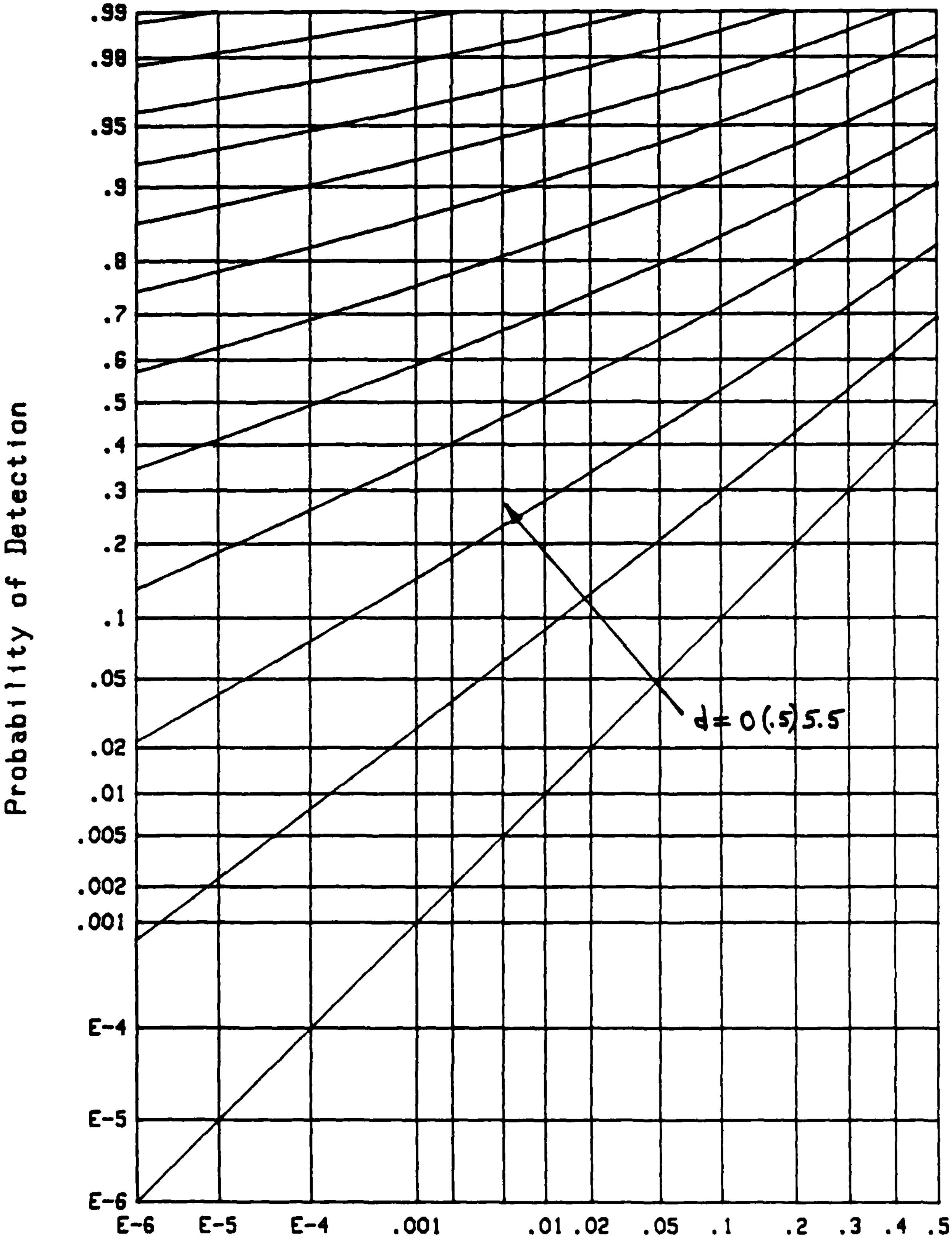

Figure 14. Probability of False Alarm

ROC for N = ∞; Extreme

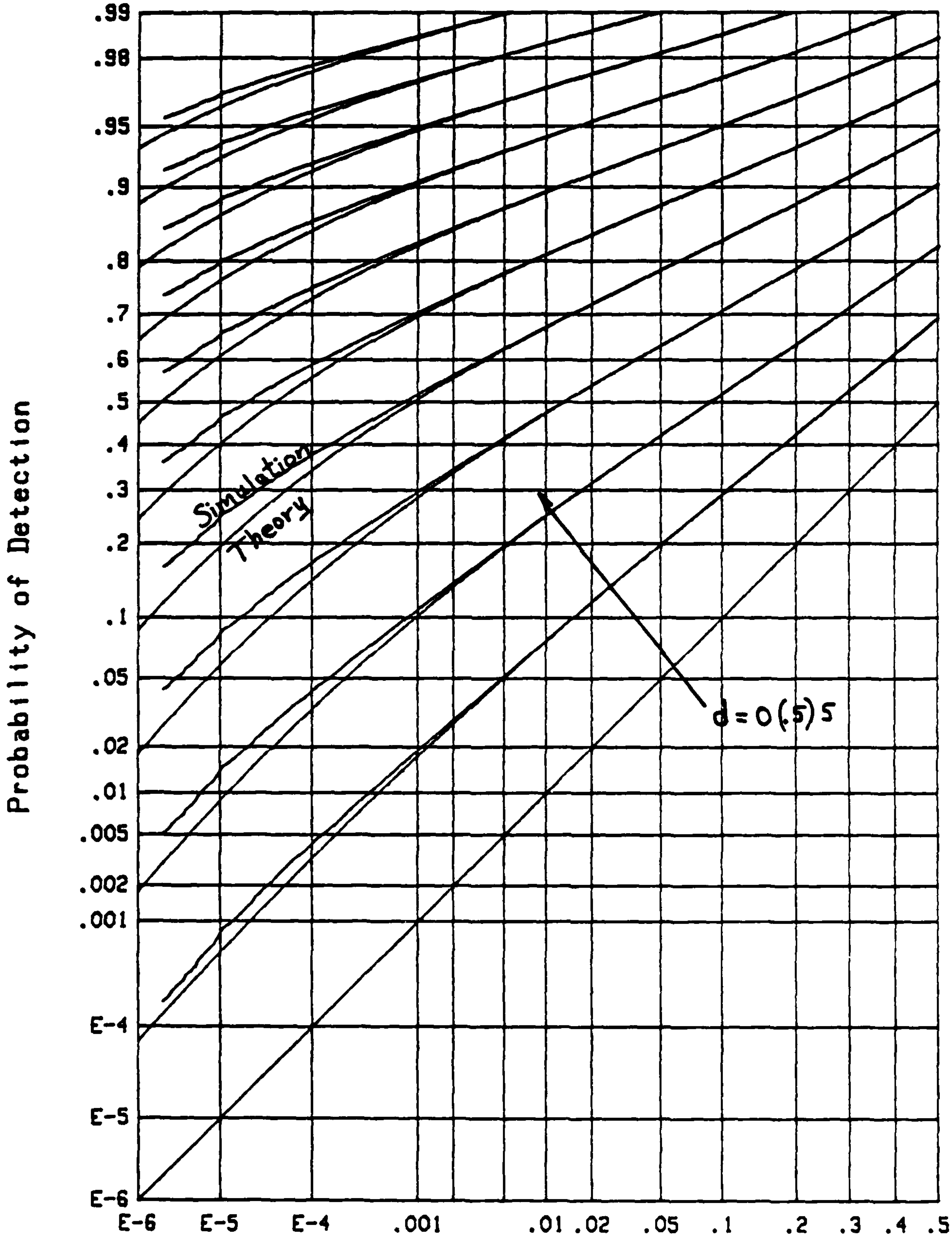

Figure 15. Probability of False Alarm
ROC for N = 64; Extreme

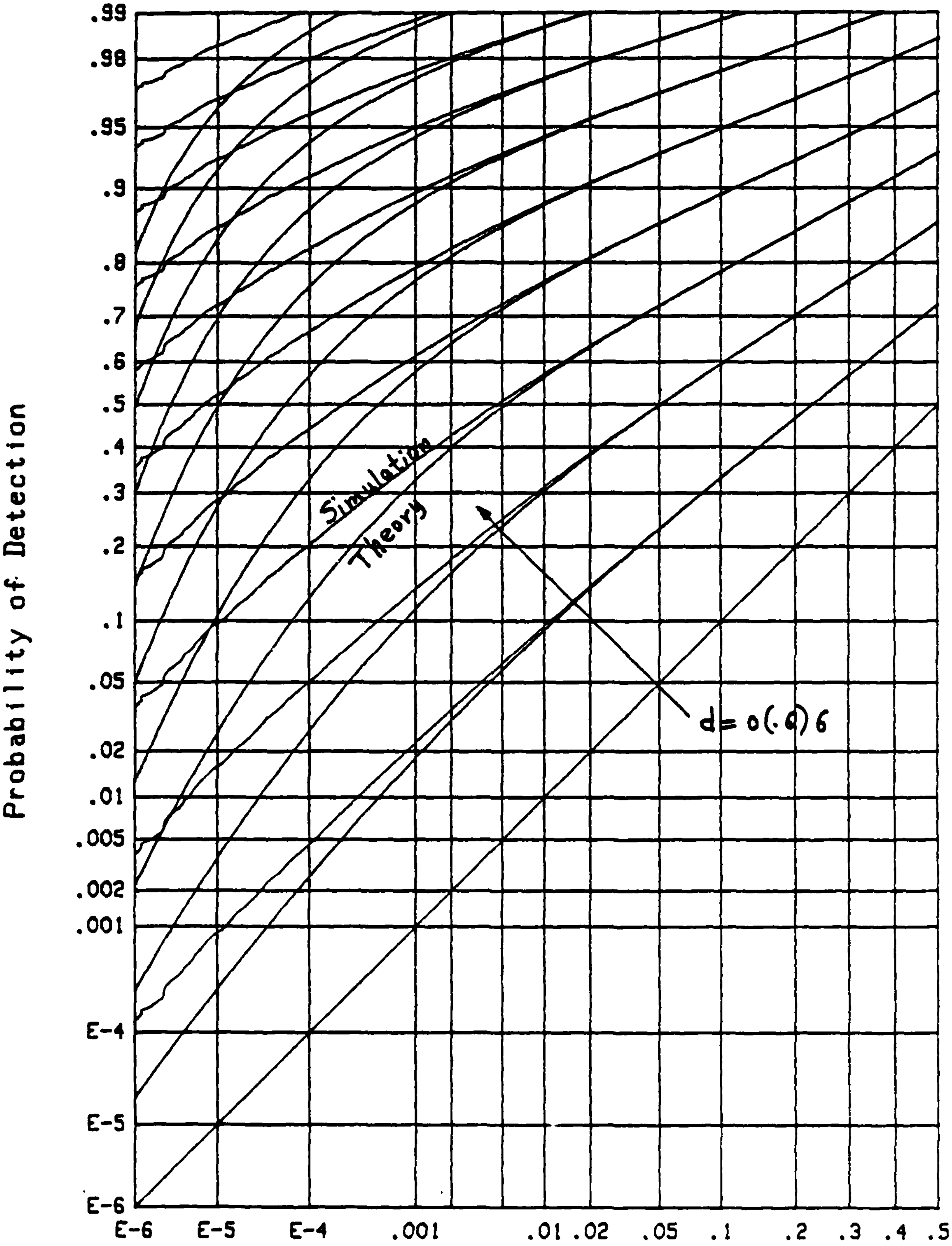

Figure 16. Probability of False Alarm
ROC for N = 32; Extreme

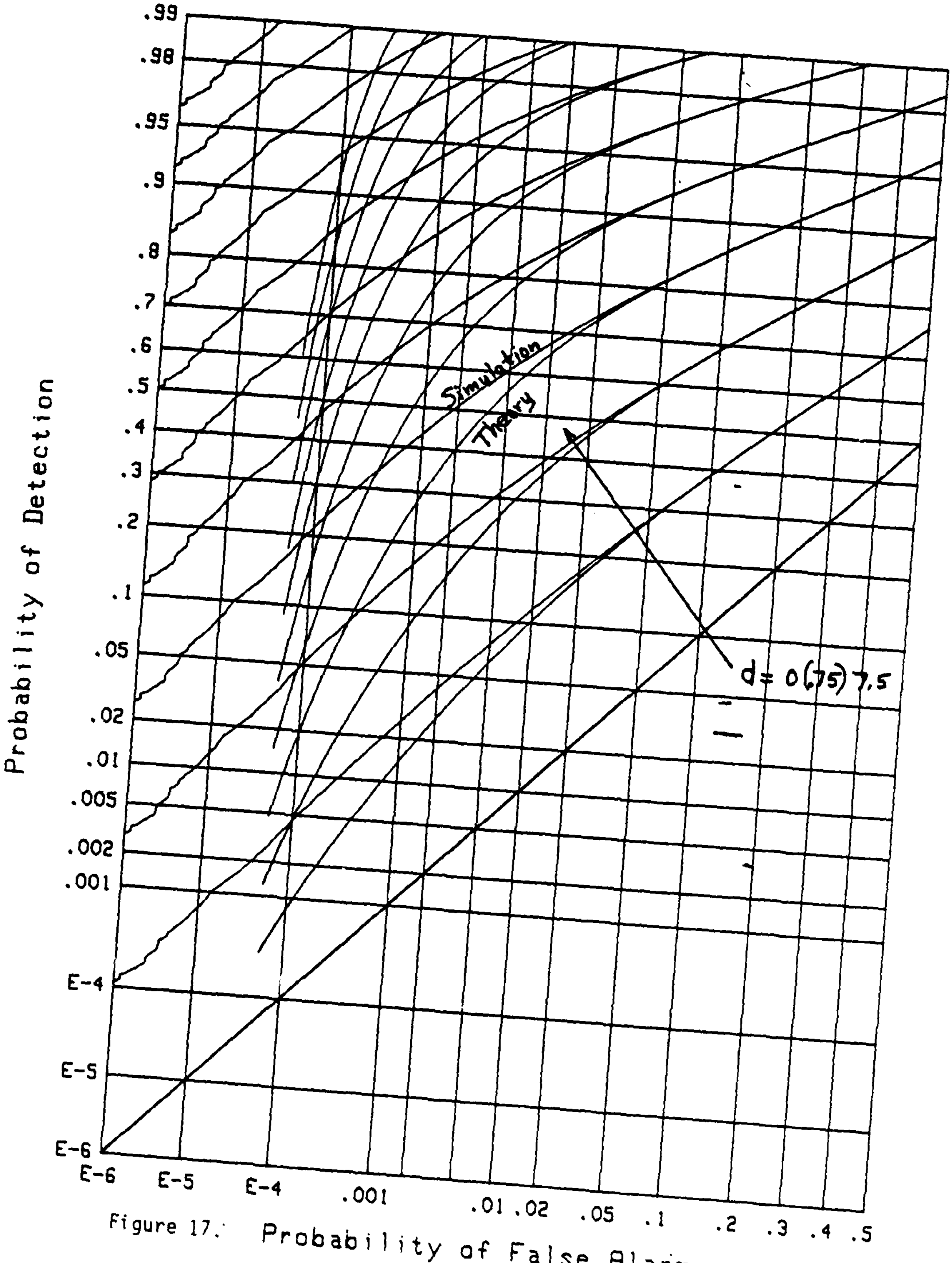

Figure 17. ROC for N = 16; Extreme

CONCLUSION

The performance of the normalizer with a Gaussian input is capable of exact analysis in terms of integrals which are readily evaluated via recursions. The main reason that this fortuitous situation obtains is the statistical independence of the sample mean and sample standard deviation for Gaussian random variables. However, for other inputs to the normalizer, these sample statistics are highly correlated with each other and create an untractable analysis problem.

An acceptable alternative in this latter case is simulation with a large number of trials. Here 8.4 million trials were employed, which allowed for estimation of tail probabilities in the 1E-6 range. If the false alarm probability could be evaluated theoretically, then simulation would only need to be conducted for the detection probability P_D. And if P_D were of interest only in the range (.5,.99) say, then as few as 10,000 trials would suffice for a decent estimate. However, it appears that, in general, even the analysis for the false alarm probability involves some unmanageable statistical relations.

APPENDIX A. WEIBULL VARIATES

The exceedance distribution function of a general Weibull random variable x is given by [8; p. 52]

$$Q_x(u) = \text{Prob}(x > u) = \exp\left[-\left(\frac{u}{a}\right)^{1/b}\right] \quad \text{for } u > 0;\ a > 0,\ b > 0\ . \qquad \text{(A-1)}$$

The corresponding probability density function of x is

$$p_x(u) = -Q_x'(u) = \frac{u^{\frac{1}{b}-1}}{a^{\frac{1}{b}}\, b} \exp\left[-\left(\frac{u}{a}\right)^{1/b}\right] \quad \text{for } u > 0\ . \qquad \text{(A-2)}$$

The υ-th moment of x is

$$\overline{x^\upsilon} = \int du\, u^\upsilon\, p_x(u) = a^\upsilon\, \Gamma(1 + b\upsilon) \quad \text{for } \upsilon > -1/b\ . \qquad \text{(A-3)}$$

The characteristic function of x is not available in closed form, for general b; however

$$\overline{\exp(i\xi x)} = (1 - i\xi a)^{-1} \quad \text{for } b = 1\ . \qquad \text{(A-4)}$$

The normalized cumulants of x, for general b, are independent of a; however, they do not approach zero as either $b \to 0$ or $b \to \infty$. Therefore x does not tend to Gaussian as the shape parameter b is changed.

LOG-DISTORTED WEIBULL VARIATE

As indicated in (1) and (15), we are interested in the log-distorted random variable

$$y = \ln x \ , \tag{A-5}$$

where x is a Weibull variate with probability density function (A-2). The exceedance distribution function of y is

$$Q_y(u) = \text{Prob}(y > u) = \text{Prob}(\ln x > u) = \text{Prob}(x > \exp(u)) =$$

$$= Q_x(\exp(u)) = \exp\left[-\frac{\exp(u/b)}{a^{1/b}}\right] \quad \text{for all } u \ , \tag{A-6}$$

where (A-1) was employed. The corresponding probability density function of random variable y is

$$p_y(u) = -Q_y'(u) = \frac{1}{b\ a^{1/b}} \exp\left[\frac{u}{b} - \frac{\exp(u/b)}{a^{1/b}}\right] \quad \text{for all } u \ , \tag{A-7}$$

which is a form of the probability density function for extreme values; see [4; (14.65)]. We will refer to (A-7) as an extreme value probability density function here.

The characteristic function of random variable y is

$$f_y(i\xi) = \overline{\exp(i\xi y)} = \overline{\exp(i\xi \ln x)} = \overline{x^{i\xi}} =$$

$$= a^{i\xi}\, \Gamma(1 + i\xi b) \,, \qquad \text{(A-8)}$$

the last step by use of (A-3); this is a generalization of [4; page 344, exercise 14.4]. The actual numerical evaluation of the characteristic function in (A-8) for real ξ is best accomplished by employing (A-7):

$$f_y(i\xi) = \overline{\exp(i\xi y)} = \int du \exp(i\xi u)\, p_y(u) =$$

$$= \frac{1}{b\, a^{1/b}} \int_{-\infty}^{+\infty} du \exp(i\xi u) \exp\left[\frac{u}{b} - \frac{\exp(u/b)}{a^{1/b}}\right] \,. \qquad \text{(A-9)}$$

This can be efficiently and accurately evaluated by use of a fast Fourier transform; the integrand decays very rapidly as $u \to \pm\infty$.

In anticipation of getting the cumulants of random variable y, we have from (A-8),

$$\ln f_y(i\xi) = i\xi \ln a + \ln \Gamma(1 + i\xi b) \,. \qquad \text{(A-10)}$$

Now from [5: (6.1.33) and section 23.2] ,

$$\ln \Gamma(1+z) = -\gamma z + \sum_{n=2}^{\infty} (-1)^n \zeta(n)\, z^n/n \,, \tag{A-11}$$

where γ = .57721 is Euler's constant and

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \,. \tag{A-12}$$

In particular, $\zeta(2) = \pi^2/6$.

There then follows, from (A-10) and (A-11), the cumulants of random variable y as

$$\chi_y(n) = \left\{ \begin{array}{ll} \ln a - b\gamma & \text{for } n=1 \\ & \\ (-1)^n \zeta(n)(n-1)!\, b^n & \text{for } n \geq 2 \end{array} \right\} . \tag{A-13}$$

In particular, the variance of y is $\chi_y(2) = b^2 \pi^2/6$. For $n \geq 2$, the normalized cumulant of y is

$$\frac{\chi_y(n)}{[\chi_y(2)]^{n/2}} = \left(-\frac{\sqrt{6}}{\pi}\right)^n \zeta(n)(n-1)! \,, \tag{A-14}$$

which is independent of both a and b; thus random variable y does not approach Gaussian as a and/or b approach any limits whatsoever. These results generalize [4; page 344, exercise 14.4].

NORMALIZED LOG-DISTORTED WEIBULL VARIATE

If a=b=1 in (7), then $x_n = w_n$, and it then follows from (16) and (1) that $\tilde{v}_n = \ln w_n = \ln x_n = y_n$. In this case, we can use (A-6) with a=b=1 to obtain the exceedance distribution function of $\tilde{v}_n$ as

$$Q_{\tilde{v}}(u) = \exp\left[-\exp(u)\right] \quad \text{for all } u . \tag{A-15}$$

Additionally, there follows from (A-13)

$$\mu(\tilde{v}) = \chi_y(1) = -\gamma = -.57721 ,$$

$$\sigma(\tilde{v}) = \left(\chi_y(2)\right)^{1/2} = \pi/\sqrt{6} . \tag{A-16}$$

We are now in position to determine the exceedance distribution function of the normalized log-distorted Weibull random variable v_n defined in (17), namely,

$$Q_v(u) = \text{Prob}(v > u) = \text{Prob}\left(\tilde{v} > \mu(\tilde{v}) + \sigma(\tilde{v})u\right) =$$

$$= Q_v\left(\mu(\tilde{v}) + \sigma(\tilde{v})u\right) = \exp\left[-\exp(-\gamma + \frac{\pi}{\sqrt{6}}\; u)\right] \quad \text{for all } u . \tag{A-17}$$

LOG-NORMAL VARIATES

For completeness, we list here the υ-th moment of log-normal variate x with probability density function as given by (14) and (12):

$$\overline{x^{\upsilon}} = \int du\ u^{\upsilon}\ p_x(u) = \frac{1}{b}\int_0^{\infty} \frac{du}{u}\ u^{\upsilon}\ \phi\left(\frac{\ln(a) - \ln(u)}{b}\right) =$$

$$= \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{+\infty} dt\ \exp\left[\upsilon t - \frac{1}{2}\left(\frac{t - \ln(a)}{b}\right)^2\right] = \exp\left[\upsilon \ln a + \frac{1}{2}\upsilon^2 b^2\right] . \tag{A-18}$$

APPENDIX B. INDEPENDENCE OF SAMPLE MEAN AND SAMPLE VARIANCE FOR GAUSSIAN RANDOM VARIABLES

Let $\{x_n\}_1^N$ be independent identically-distributed Gaussian random variables with mean and variance

$$\overline{x_n} = \mu , \quad \overline{(x_n - \mu)^2} = \sigma^2 \quad \text{for all } n . \tag{B-1}$$

Define sample mean

$$m = \frac{1}{N} \sum_{n=1}^{N} x_n , \tag{B-2}$$

and sample variance

$$v = g \sum_{n=1}^{N} (x_n - m)^2 , \tag{B-3}$$

where scale factor $g = 1/N$ or $1/(N-1)$ typically. ($N \geq 2$ required.)

We have, in vector notation

$$m = \frac{1}{N} 1^T X = \frac{1}{N}[1 \ 1 \ \dots \ 1][x_1 \ x_2 \ \dots \ x_N]^T , \tag{B-4}$$

and

$$v = g \left[\sum_{n=1}^{N} x_n^2 - m^2 N \right] = g \left[X^T X - \frac{1}{N} X^T 1 \ 1^T X \right] = g \ X^T Q X , \tag{B-5}$$

where

$$Q = I - \frac{1}{N} 1\, 1^T . \tag{B-6}$$

The joint characteristic function of m and v is

$$f(\xi,\phi) = \overline{\exp(i\xi m + i\phi v)} =$$

$$= \overline{\exp\left[i\xi\frac{1}{N} 1^T X + i\phi g\, X^T Q X\right]} . \tag{B-7}$$

Now the joint probability density function of vector X is

$$p(X) = \prod_{n=1}^{N} (\sqrt{2\pi}\,\sigma)^{-1} \exp\left[-\frac{(x_n - \mu)^2}{2\,\sigma^2}\right] =$$

$$= \left(2\pi\,\sigma^2\right)^{-N/2} \exp\left[-\frac{1}{2\sigma^2}\left(X^T X - 2\mu\, 1^T X + \mu^2 N\right)\right] . \tag{B-8}$$

Therefore

$$f(\xi,\phi) = (2\pi\,\sigma^2)^{-N/2} \int dX \exp\left[-\frac{1}{2\sigma^2}\left(X^T X - 2\mu\, 1^T X + \mu^2 N\right) + \right.$$

$$\left. + i\frac{\xi}{N} 1^T X + i\phi g\, X^T Q X\right] . \tag{B-9}$$

Now we use [6; (B-1)]

$$\int dX \exp\left[-\frac{1}{2} X^T M X + L^T X\right] = \left[\frac{(2\pi)^N}{\det M}\right]^{1/2} \exp\left[\frac{1}{2} L^T M^{-1} L\right] \tag{B-10}$$

with identifications

$$M = \frac{1}{\sigma^2} I - i2\phi g\, Q , \qquad L = \left(\frac{\mu}{\sigma^2} + i\frac{\xi}{N}\right) 1 . \tag{B-11}$$

Using the definition of Q in (B-6), there follows

$$M = \left(\frac{1}{\sigma^2} - i2\phi g\right) I + \frac{i2\phi g}{N} 1\, 1^T . \tag{B-12}$$

Now from [6; (21) and (22)],

$$\det M = \frac{1}{\sigma^{2N}} (1 - i2\sigma^2 \phi g)^{N-1} ,$$

$$M^{-1} = \frac{\sigma^2}{1 - i2\sigma^2 \phi g} \left[I - \frac{i2\phi g \sigma^2}{N} 1\, 1^T \right] . \tag{B-13}$$

There follows

$$L^T M^{-1} L = N\, \sigma^2 \left(\frac{\mu}{\sigma^2} + i\, \frac{\xi}{N}\right)^2 \tag{B-14}$$

and

$$f(\xi,\phi) = \exp\left[i\xi\mu - \frac{1}{2}\, \xi^2\, \frac{\sigma^2}{N} \right] \left(1 - i2\sigma^2 g\phi\right)^{\frac{1-N}{2}} . \tag{B-15}$$

Since this joint characteristic function factors, it follows that sample statistics m and v are statistically independent. Also the probability density functions are obviously

$$p_m(u) = \frac{1}{\sqrt{2\pi}\, \sigma/\sqrt{N}} \exp\left[- \frac{(u - \mu)^2}{2\, \sigma^2/N} \right] \quad \text{for all } u \tag{B-16}$$

and

$$p_v(u) = \frac{u^{\frac{N-3}{2}} \exp\left(\frac{-u}{2\sigma^2 g}\right)}{\Gamma\left(\frac{N-1}{2}\right) \left(2\sigma^2 g\right)^{\frac{N-1}{2}}} \quad \text{for } u > 0 . \tag{B-17}$$

Thus sample mean m is Gaussian with mean μ and variance σ^2/N; while sample variance v is chi-squared of N-1 degrees of freedom with mean $\sigma^2(N-1)g$ and variance $2\sigma^4(N-1)g^2$. For the typical choice of gain $g = 1/(N-1)$, this implies that v has mean σ^2 and variance $2\sigma^4/(N-1)$, and therefore

$$\lim_{N\to\infty} p_v(u) = \delta(u - \sigma^2) \ . \tag{B-18}$$

The sample standard deviation

$$s = \sqrt{v} \tag{B-19}$$

has probability density function

$$p_s(u) = 2\,u\,p_v(u^2) = \frac{2\,u^{N-2}\exp(-u^2/\beta)}{\Gamma\left(\frac{N-1}{2}\right)\beta^{\frac{N-1}{2}}} \quad \text{for } u > 0 \tag{B-20}$$

where

$$\beta = \frac{2\sigma^2}{N-1} \quad \text{for } g = 1/(N-1) \ . \tag{B-21}$$

The k-th moment of s is

$$\overline{s^k} = \int du\, u^k\, p_s(u) = \int_0^\infty du\, \frac{2\,u^{N-2+k}\exp(-u^2/\beta)}{\Gamma\left(\frac{N-1}{2}\right)\beta^{\frac{N-1}{2}}} = \frac{\Gamma\left(\frac{N+k-1}{2}\right)\beta^{k/2}}{\Gamma\left(\frac{N-1}{2}\right)} \ . \tag{B-22}$$

In particular,

$$\overline{s^2} = \sigma^2 \ , \tag{B-23}$$

and [5; (6.1.47)]

$$\bar{s} = \sigma \left(\frac{2}{N-1}\right)^{1/2} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} =$$

$$= \sigma \left(\frac{2}{N-1}\right)^{1/2} \left(\frac{N - \frac{3}{2}}{2}\right)^{1/2} \left[1 + \frac{1/16}{\left(N - \frac{3}{2}\right)^2} + O\left(N - \frac{3}{2}\right)^{-3}\right] =$$

$$= \sigma \left[1 - \frac{1/4}{N - \frac{9}{8}} + O\left(N - \frac{9}{8}\right)^{-3}\right] \left[1 + \frac{1/16}{\left(N - \frac{3}{2}\right)^2} + O\left(N - \frac{3}{2}\right)^{-3}\right] =$$

$$= \sigma \left[1 - \frac{1/4}{N - \frac{9}{8}} + \frac{1/16}{\left(N - \frac{3}{2}\right)^2} + O(N^{-3})\right] =$$

$$= \sigma \left[1 - \frac{1/4}{N - \frac{7}{8}} + O(N^{-3})\right] \quad \text{as } N \to \infty \; . \qquad \text{(B-24)}$$

APPENDIX C. PROBABILITY RECURSIONS FOR GAUSSIAN CASE

The detection and false alarm probabilities are given in integral form in (38) and (41). These integrals have already been encountered in [1; appendix E], and evaluated in a recursive fashion. We will modify those results somewhat, in order to better suit the current forms.

First, we have, from (41) and (39),

$$P_F = \int_0^\infty dw \, \frac{w^{N-2}\exp(-w^2/2)}{2^{\frac{N-3}{2}}\,\Gamma\left(\frac{N-1}{2}\right)} \, \Phi(-T_1'w) \equiv P_F(N,T_1') \, . \tag{C-1}$$

Define

$$x_1 = \left(1 + T_1'^2\right)^{-1} , \quad \text{where } T_1' = T\sqrt{\frac{N}{N^2 - 1}} \, . \tag{C-2}$$

Then from [1; (E-17)], using identifications (that is, replacements from there to here)

$$r \rightarrow \frac{T_1'}{\left(1 + T_1'^2\right)^{1/2}} , \quad K \rightarrow N-2 , \tag{C-3}$$

there follows the simple result

$$P_F(N,T_1') = \frac{1}{2} - \frac{1}{\pi}\mathrm{atn}(T_1') - \frac{1}{\pi} T_1' x_1 \sum_{k=0}^{\frac{N}{2}-2} b_k x_1^k \quad \text{for } N=2,4,6,\ldots, \tag{C-4}$$

where

$$b_0 = 1 , \quad b_k = b_{k-1} \frac{k}{k+\frac{1}{2}} \quad \text{for } k \geq 1 . \tag{C-5}$$

A program for this false alarm probability is given in appendix E under the name FNPf246, where T_1' is represented by variable Tp.

Also,

$$P_F(N,T_1') = \frac{1}{2} - \frac{1}{2} T_1'\sqrt{x_1} \sum_{k=0}^{\frac{N-3}{2}} a_k x_1^k \quad \text{for } N=3,5,7,\ldots, \tag{C-6}$$

where

$$a_0 = 1 , \quad a_k = a_{k-1} \frac{k-\frac{1}{2}}{k} \quad \text{for } k \geq 1 . \tag{C-7}$$

These results are very tractable and efficient forms for recursive computer evaluation. A program for (C-6) is given in appendix E under the name FNPf357, where T_1' is represented by variable Tp.

The current form for detection probability $P_D = P_D(N,T_r',d_r')$ in (38) is identical to [1; (E-1)] if we make replacements

$$d_T \to d'_r\,, \qquad r \to \frac{T'_r}{\left(1 + T'^2_r\right)^{1/2}}\,, \qquad K \to N-2\,. \tag{C-8}$$

(The curves in [1] are not directly applicable here because they employed the fundamental parameter $d_T \to d'_r$, which is $d/\left(r^2 + 1/N\right)^{1/2}$ here; however, the recursions derived there are immediately useable.) We can then use [1; (E-8)] to develop an expression for P_D, in terms of the auxiliary sequence $\{g(K)\}$ defined in [1; (E-7)]. In particular, [1; (E-9)] yields, with

$$x_r = \left(1 + T'^2_r\right)^{-1}\,, \tag{C-9}$$

the result

$$g(0) = T'_r\sqrt{x_r}\,\exp\left(-\frac{1}{2}\,d'^2_r x_r\right)\,\Phi\left(d'_r T'_r \sqrt{x_r}\right)\,; \tag{C-10}$$

[1; (E-13)] yields

$$g(1) = T'_r x_r\left[\frac{1}{\pi}\exp\left(-d'^2_r/2\right) + \left(\frac{2}{\pi}\right)^{1/2} d'_r\, g(0)\right]\,; \tag{C-11}$$

and [1; (E-12)] yields

$$g(K) = x_r\left[h(K)g(K-1) + \frac{K-1}{K}\,g(K-2)\right] \quad \text{for } K \geq 2\,, \tag{C-12}$$

with definition

$$h(K) = \frac{1}{\sqrt{2}}\,T'_r d'_r\,\frac{\Gamma\left(\frac{K+1}{2}\right)}{\Gamma\left(\frac{K}{2}+1\right)}\,. \tag{C-13}$$

Then we can also use

$$h(0) = T_r' \, d_r' \left(\frac{\pi}{2}\right)^{1/2} , \quad h(1) = T_r' \, d_r' \left(\frac{2}{\pi}\right)^{1/2} ,$$

$$h(K) = h(K-2) \, \frac{K-1}{K} \quad \text{for } K \geq 2 . \tag{C-14}$$

Finally, $P_D = P_D(N,T_r',d_r')$ is given by [1; (E-8)] as

$$P_D = \begin{cases} \Phi(d_r') - \displaystyle\sum_{\substack{K=0 \\ K \text{ even}}}^{N-3} g(K) & \text{for } N=3,5,7, \ldots \\ \\ P_{D2} - \displaystyle\sum_{\substack{K=1 \\ K \text{ odd}}}^{N-3} g(K) & \text{for } N=4,6,8, \ldots \end{cases} , \tag{C-15}$$

where P_{D2} is the value of detection probability P_D for $N = 2$. Observe that the input parameters to P_D are N,T_r',d_r', rather than the four fundamental parameters in (40); that is, N,T,d,r are collapsed into N,T_r',d_r' according to (39). Programs for (C-15) are furnished in appendix E under the names FNPd357 and FNPd246, respectively, where T_r' and d_r' are represented by variables Tp and Dp.

The quantity P_{D2} in (C-15) is evaluated according to the method in [1; appendix F]; an error tolerance and maximum number of terms must also be specified to terminate the infinite sum given by [1; (F-2)].

APPENDIX D. χ-APPROXIMATION FOR RANDOM VARIABLE t

Suppose we assume that the random variable t in (46) is a multiple of a χ-variate with K degrees of freedom; then its probability density function is [7; pages 5-7 for $\nu = 1/2$]

$$p_t(u) = \frac{u^{K-1} \exp\left(-u^2/\left(2A^2\right)\right)}{A^K \, 2^{\frac{K}{2}-1} \, \Gamma\left(\frac{K}{2}\right)} \quad \text{for } u > 0 \; . \tag{D-1}$$

Then the ν-th moment of random variable t is

$$\overline{t^\nu} = \int du \; u^\nu \; p_t(u) = \frac{2^{\nu/2} A^\nu \, \Gamma\left(\frac{K+\nu}{2}\right)}{\Gamma\left(\frac{K}{2}\right)} \; , \tag{D-2}$$

and in particular

$$\bar{t} = A \, \frac{\sqrt{2} \, \Gamma\left(\frac{K+1}{2}\right)}{\Gamma\left(\frac{K}{2}\right)} \; , \quad \overline{t^2} = A^2 K \; . \tag{D-3}$$

Then the ratio

$$R = \frac{\bar{t}}{\sqrt{\overline{t^2}}} = \frac{\Gamma\left(\frac{K+1}{2}\right)}{\sqrt{\frac{K}{2}} \, \Gamma\left(\frac{K}{2}\right)} = 1 - \frac{1}{4K+\frac{1}{2}} + O(K^{-3}) \; , \tag{D-4}$$

where the last result uses the development in (B-24) with N replaced by K+1.

Given a value for ratio R on the left side of (D-4), K can be solved for uniquely, since the ratio involving gamma functions increases monotonically from 0 to 1 as K goes from 0 to $+\infty$. In fact, to a good approximation for large K, the last part of (D-4) gives

$$K \cong \frac{1}{4(1-R)} - \frac{1}{8} \ . \qquad \text{(D-5)}$$

Here we are allowing K in probability density function p_t in (D-1) to be arbitrary, that is, not limited to integer values. Then we can solve for the required value of A according to (D-3), as $A^2 = \overline{t^2}/K$. This procedure fits the assumed probability density function form in (D-1) to specified values of the first two moments of t given by (D-3), as given by simulation results (53)-(55).

If we now employ the χ-approximate probability density function for t given by (D-1) in detection probability result (45), along with (49), we obtain

$$P_D = \int_0^\infty du \, \exp\left[-\exp\left(-\gamma + \frac{\pi}{\sqrt{6}} \frac{u - d}{r}\right)\right] \frac{u^{K-1} \exp\left(-u^2/\left(2A^2\right)\right)}{A^K \, 2^{\frac{K}{2} - 1} \, \Gamma\left(\frac{K}{2}\right)} =$$

$$= \left[2^{\frac{K}{2} - 1} \, \Gamma\left(\frac{K}{2}\right)\right]^{-1} \int_0^\infty dx \, x^{K-1} \exp\left[-x^2/2 - \exp(h_1 + h_2 \, x)\right] \ , \qquad \text{(D-6)}$$

where constants

$$h_1 = -\gamma - \frac{\pi}{\sqrt{6}} \frac{d}{r} \ , \qquad h_2 = \frac{\pi}{\sqrt{6}} \frac{A}{r} \ . \qquad \text{(D-7)}$$

The numerical evaluation of (D-6) was undertaken for N = 16, and is discussed in the Graphical Results section of this report.

APPENDIX E. PROGRAMS

In this appendix, four programs are listed. They are written in BASIC for the Hewlett-Packard 9000 Model 520 Desk Top Computer. Their titles are

EDF - Gaussian,

EDF - Extreme,

Simulation-Extreme,

Plot-Simulation.

The first one computes the exceedance distribution function for a Gaussian input to the normalizer of figure 1, for N=16 (line 10) and for d = 0(1)12 (lines 960-970). This program is heavily based on the results of appendix C.

The second program computes the approximate exceedance distribution function for a log-distorted Weibull input to the normalizer of figure 1, for N=16 (line 10), r=1 (line 20), and d = 0(.75)7.5 (line 1070). It is based on numerical integration of (51) via Simpson's rule.

The third program simulates the normalizer output (28) and (24) for a log-distorted Weibull input, for N=16 (line 10), d = 0(.75)7.5 (line 20), r= 1 (line 30), and 2^{23} = 8.4 million trials (line 40). The range of

values in z is (-15,5), which is divided into 1000 bins; see lines 60, 90, 100. The resultant histogram is then summed on the upper tail to yield the exceedance distribution function. The fourth program plots these simulation results for the exceedance distribution function vs threshold T.

Table E-1. EDF - Gaussian

```
10    Ns=16                                       ! NUMBER OF SAMPLES
20    X1=-7                                       ! THRESHOLD
30    X2=7                                        !  LIMITS
40    DIM A$[30],B$[30]
50    DIM Xlabel$(1:30),Ylabel$(1:30)
60    DIM Xcoord(1:30),Ycoord(1:30)
70    DIM Xgrid(1:30),Ygrid(1:30)
80    DOUBLE Lx,Ly,Nx,Ny,I,Ns,It
90    !
100   A$="Threshold"
110   B$="Exceedance Probability"
120   !
130   Lx=15
140   REDIM Xlabel$(1:Lx),Xcoord(1:Lx)
150   DATA -7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7
160   READ Xlabel$(*)
170   DATA -7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7
180   READ Xcoord(*)
190   !
200   Ly=27
210   REDIM Ylabel$(1:Ly),Ycoord(1:Ly)
220   DATA E-6,E-5,E-4,.001,.002,.005,.01,.02,.05
230   DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
240   DATA .995,.998,.999,.9999,.99999,.999999
250   READ Ylabel$(*)
260   DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05
270   DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
280   DATA .995,.998,.999,.9999,.99999,.999999
290   READ Ycoord(*)
300   !
310   Nx=15
320   REDIM Xgrid(1:Nx)
330   DATA -7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7
340   READ Xgrid(*)
350   !
360   Ny=27
370   REDIM Ygrid(1:Ny)
380   DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05
390   DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
400   DATA .995,.998,.999,.9999,.99999,.999999
410   READ Ygrid(*)
420   !
430   FOR I=1 TO Ly
440   Ycoord(I)=FNInvphi(Ycoord(I))
450   NEXT I
460   FOR I=1 TO Ny
470   Ygrid(I)=FNInvphi(Ygrid(I))
480   NEXT I
490   Y1=Ygrid(1)
500   Y2=Ygrid(Ny)
```

```
510     GINIT 180./240.                          !  VERTICAL PAPER
520     PLOTTER IS 505,"HPGL"
530     PRINTER IS 505
540     LIMIT PLOTTER 505,0.,180.,0.,240.        !  1 GDU = 2 mm
550     VIEWPORT 20.,120.,19.,132.
560   ! VIEWPORT 22.,85.,59.,122.                !  TOP OF PAPER
570   ! VIEWPORT 22.,85.,19.,62.                 !  BOTTOM OF PAPER
580     WINDOW X1,X2,Y1,Y2
590   !   PRINT "VS2"
600     FOR I=1 TO Nx
610     MOVE Xgrid(I),Y1
620     DRAW Xgrid(I),Y2
630     NEXT I
640     FOR I=1 TO Ny
650     MOVE X1,Ygrid(I)
660     DRAW X2,Ygrid(I)
670     NEXT I
680     PENUP
690     LDIR 0
700     CSIZE 2.3,.5
710     LORG 5
720     Y=Y1-(Y2-Y1)*.02
730     FOR I=1 TO Lx
740     MOVE Xcoord(I),Y
750     LABEL Xlabel$(I)
760     NEXT I
770     CSIZE 3.,.5
780     MOVE .5*(X1+X2),Y1-.06*(Y2-Y1)
790     LABEL A$
800     CSIZE 2.3,.5
810     LORG 8
820     X=X1-(X2-X1)*.01
830     FOR I=1 TO Ly
840     MOVE X,Ycoord(I)
850     LABEL Ylabel$(I)
860     NEXT I
870     LDIR PI/2.
880     CSIZE 3.,.5
890     LORG 5
900     MOVE X1-.15*(X2-X1),.5*(Y1+Y2)
910     LABEL B$
920       PRINT "VS36"
```

```
930    Dx=(X2-X1)/100.
940    F1=SQR(Ns/(Ns+1))
950    F2=SQR(Ns/(Ns*Ns-1))
960    FOR I=0 TO 12
970    Ds=I                              !  DEFLECTION d
980    Dp=Ds*F1                          !  d'
990    FOR It=0 TO 100
1000   T=X1+Dx*It                        !  THRESHOLD T
1010   Tp=T*F2                           !  T'
1020   IF Dp>0. THEN 1080
1030   IF Ns MODULO 2=0 THEN 1060
1040   Pd=FNPf357(Ns,Tp)
1050   GOTO 1120
1060   Pd=FNPf246(Ns,Tp)
1070   GOTO 1120
1080   IF Ns MODULO 2=0 THEN 1110
1090   Pd=FNPd357(Ns,Tp,Dp)
1100   GOTO 1120
1110   Pd=FNPd246(Ns,Tp,Dp)
1120   IF Pd<=0. THEN 1160
1130   IF Pd>=1. THEN 1160
1140   Y=FNInvphi(Pd)
1150   PLOT T,Y
1160   NEXT It
1170   PENUP
1180   NEXT I
1190   PAUSE
1200   PRINTER IS CRT
1210   PLOTTER 505 IS TERMINATED
1220   END
1230   !
1240   DEF FNInvphi(X)        !  AMS 55, 26.2.23
1250   IF X=.5 THEN RETURN 0.
1260   P=MIN(X,1.-X)
1270   T=-LOG(P)
1280   T=SQR(T+T)
1290   P=1.+T*(1.432788+T*(.189269+T*.001308))
1300   P=T-(2.515517+T*(.802853+T*.010328))/P
1310   IF X<.5 THEN P=-P
1320   RETURN P
1330   FNEND
1340   !
1350   DEF FNPf246(DOUBLE N,REAL Tp)        !  N=2,4,6,...
1360   DOUBLE Ks                            !  INTEGER
1370   Pf=.5-ATN(Tp)/PI
1380   IF N=2 THEN RETURN Pf
1390   X=1./(1.+Tp*Tp)
1400   S=B=1.
1410   FOR Ks=1 TO N/2-2
1420   B=B*X*Ks/(Ks+.5)
1430   S=S+B
1440   NEXT Ks
1450   Pf=Pf-Tp*X*S/PI
1460   RETURN Pf
1470   FNEND
1480   !
```

```
1490    DEF FNPf357(DOUBLE N,REAL Tp)        !  N=3,5,7,...
1500    DOUBLE Ks                            !  INTEGER
1510    X=1./(1.+Tp*Tp)
1520    S=Ax=1.
1530    FOR Ks=1 TO (N-3)/2
1540    Ax=Ax*X*(Ks-.5)/Ks
1550    S=S+Ax
1560    NEXT Ks
1570    Pf=.5*(1.-Tp*SQR(X)*S)
1580    RETURN Pf
1590    FNEND
1600    !
1610    DEF FNPd246(DOUBLE N,REAL Tp,Dp)     !  N=2,4,6,...
1620    Error=1.E-15                         !  Tolerance in Pd for N=2
1630    Nterms=500                           !  Number of terms for N=2
1640    DOUBLE Ks                            !  INTEGER
1650    X=1./(1.+Tp*Tp)
1660    Rk=SQR(X)
1670    Tsq=Tp*Rk
1680    Dsq=Dp*Dp
1690    R=SQR(2./PI)
1700    A1=A0=EXP(-.5*Dsq)/PI
1710    A=A1*Dp/R
1720    B1=.5*PI-ATN(Tp)
1730    B=Rk
1740    Pd=A1*B1+A*B
1750    FOR Ks=2 TO Nterms
1760    F=FLT(Ks-1)
1770    T=Dsq*A1/F
1780    A1=A
1790    A=T
1800    Rk=Rk*Tsq
1810    T=(Rk+F*B1)/Ks
1820    B1=B
1830    B=T
1840    P=A*B
1850    Pd=Pd+P
1860    IF P<=Error*Pd THEN 1890
1870    NEXT Ks
1880    PRINT "500 TERMS IN FNPd246"
1890    IF N=2 THEN RETURN Pd
1900    G1=Tsq*EXP(-.5*Dsq*X)*FNPhi(Dp*Tsq)
1910    G=Tp*X*(A0*R*Dp+G1)
1920    F1=Tp*Dp/R
1930    F=Tp*Dp*R
1940    Pd=Pd-G
1950    IF N=4 THEN RETURN Pd
1960    FOR Ks=2 TO N-3
1970    R=(Ks-1)/Ks
1980    T=F1*R
1990    F1=F
2000    F=T
2010    T=X*(F*G+R*G1)
2020    G1=G
2030    G=T
2040    IF Ks MODULO 2=1 THEN Pd=Pd-G
2050    NEXT Ks
2060    RETURN Pd
2070    FNEND
2080    !
```

```
2090    DEF FNPd357(DOUBLE N,REAL Tp,Dp)   !  N=3,5,7,...
2100    DOUBLE Ks                          !  INTEGER
2110    X=1./(1.+Tp*Tp)
2120    Tsq=Tp*SQR(X)
2130    D2=.5*Dp*Dp
2140    G=Tsq*EXP(-D2*X)*FNPhi(Dp*Tsq)
2150    Pd=FNPhi(Dp)-G
2160    IF N=3 THEN RETURN Pd
2170    R=SQR(2./PI)
2180    G1=G
2190    G=Tp*X*(EXP(-D2)/PI+R*Dp*G1)
2200    F1=Tp*Dp/R
2210    F=Tp*Dp*R
2220    FOR Ks=2 TO N-3
2230    R=(Ks-1)/Ks
2240    T=F1*R
2250    F1=F
2260    F=T
2270    T=X*(F*G+R*G1)
2280    G1=G
2290    G=T
2300    IF Ks MODULO 2=0 THEN Pd=Pd-G
2310    NEXT Ks
2320    RETURN Pd
2330    FNEND
2340    !
2350    DEF FNPhi(X)         !  HART, page 140, #5708 & #5725
2360    Y=ABS(X)*.70710678118654746
2370    SELECT Y
2380    CASE <8.
2390    P=1631.76026875371470+Y*(456.261458706092631+Y*(86.0827622119485951+Y*(
10.0648589749095425+Y*.564189586761813614)))
2400    P=3723.50798155480672+Y*(7113.66324695404987+Y*(6758.21696411048589+Y*(
4032.26701083004974+Y*P)))
2410    Q=7542.47951019347576+Y*(2968.00490148230872+Y*(317.622386304544077+Y*(
153.077710750362216+Y*(17.8394984391395565+Y))))
2420    Q=3723.50798155480654+Y*(11315.1920813544055+Y*(15802.5159944020425+Y*(
13349.3465612844574+Y*Q)))
2430    Phi=.5*EXP(-Y*Y)*P/Q
2440    CASE <26.6
2450    P=2.97886562639399289+Y*(7.40974060596474179+Y*([illegible]+Y*(
5.01904972678426746+Y*(1.27536664472996595+Y*.564189583547755074))))
2460    Q=3.36907520698275277+Y*(9.60896532719278787+Y*(17.0814407474660041+Y*(
12.0489519278551290+Y*(9.39603401623505415+Y*(2.26052352079[illegible]+
2470    Phi=.5*EXP(-Y*Y)*P/Q
2480    CASE ELSE
2490    Phi=0.
2500    END SELECT
2510    IF [illegible]0. THEN Phi=1.-Phi
2520    RETURN Phi
2530    FNEND
```

Table E-2. EDF - Extreme

```
10    Ns=16                   !  NUMBER OF SAMPLES
20    Rs=1.                   !  RATIO r
30    X1=-15
40    X2=5
50    Gamma=.577215664902
60    C1=-Gamma
70    C2=PI/SQR(6.)
80    F=SQR(.5/PI)
90    A=-8.                   !  LIMITS ON
100   B=8.                    !  INTEGRAL
110   Mean_mucv=0.
120   Var_mucv=1./Ns
130   Mean_sigcv=1.-.39/Ns
140   Var_sigcv=1.05/(Ns+1.5)
150   Rho=-.55
160   COM H1,H2
170   DIM A$[30],B$[30]
180   DIM Xlabel$(1:30),Ylabel$(1:30)
190   DIM Xcoord(1:30),Ycoord(1:30)
200   DIM Xgrid(1:30),Ygrid(1:30)
210   DOUBLE Lx,Ly,Nx,Ny,I,Ns,It
220   !
230   A$="Threshold"
240   B$="Exceedance Probability"
250   !
260   Lx=11
270   REDIM Xlabel$(1:Lx),Xcoord(1:Lx)
280   DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
290   READ Xlabel$(*)
300   DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
310   READ Xcoord(*)
320   !
330   Ly=27
340   REDIM Ylabel$(1:Ly),Ycoord(1:Ly)
350   DATA E-6,E-5,E-4,.001,.002,.005,.01,.02,.05,.1,.2,.3,.4,.5
360   DATA .6,.7,.8,.9,.95,.98,.99
370   DATA .995,.998,.999,.9999,.99999,.999999
380   READ Ylabel$(*)
390   DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05,.1,.2,.3,.4,.5
400   DATA .6,.7,.8,.9,.95,.98,.99
410   DATA .995,.998,.999,.9999,.99999,.999999
420   READ Ycoord(*)
430   !
440   Nx=11
450   REDIM Xgrid(1:Nx)
460   DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
470   READ Xgrid(*)
480   !
490   Ny=27
500   REDIM Ygrid(1:Ny)
510   DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05,.1,.2,.3,.4,.5
520   DATA .6,.7,.8,.9,.95,.98,.99
530   DATA .995,.998,.999,.9999,.99999,.999999
540   READ Ygrid(*)
550   !
```

```
560     FOR I=1 TO Ly
570     Ycoord(I)=FNInvphi(Ycoord(I))
580     NEXT I
590     FOR I=1 TO Ny
600     Ygrid(I)=FNInvphi(Ygrid(I))
610     NEXT I
620     Y1=Ygrid(1)
630     Y2=Ygrid(Ny)
640     GINIT 180./240.                         ! VERTICAL PAPER
650     PLOTTER IS 505,"HPGL"
660     PRINTER IS 505
670     LIMIT PLOTTER 505,0.,180.,0.,240.       ! 1 GDU = 2 mm
680     VIEWPORT 20.,120.,19.,132.
690   ! VIEWPORT 22.,85.,59.,122.               ! TOP OF PAPER
700   ! VIEWPORT 22.,85.,19.,62.                ! BOTTOM OF PAPER
710     WINDOW X1,X2,Y1,Y2
720   !   PRINT "VS2"
730     FOR I=1 TO Nx
740     MOVE Xgrid(I),Y1
750     DRAW Xgrid(I),Y2
760     NEXT I
770     FOR I=1 TO Ny
780     MOVE X1,Ygrid(I)
790     DRAW X2,Ygrid(I)
800     NEXT I
810     PENUP
820     CSIZE 2.3,.5
830     LORG 5
840     Y=Y1-(Y2-Y1)*.02
850     FOR I=1 TO Lx
860     MOVE Xcoord(I),Y
870     LABEL Xlabel$(I)
880     NEXT I
890     CSIZE 3.,.5
900     MOVE .5*(X1+X2),Y1-.06*(Y2-Y1)
910     LABEL A$
920     CSIZE 2.3,.5
930     LORG 8
940     X=X1-(X2-X1)*.01
950     FOR I=1 TO Ly
960     MOVE X,Ycoord(I)
970     LABEL Ylabel$(I)
980     NEXT I
990     LDIR PI/2.
1000    CSIZE 3.,.5
1010    LORG 5
1020    MOVE X1-.15*(X2-X1),.5*(Y1+Y2)
1030    LABEL B$
1040      PRINT "VS36"
1050    Dx=(X2-X1)/100.
1060    DIM F(0:100)
```

```
1070    FOR Ds=0. TO 7.5 STEP .75          !  DEFLECTION d
1080    FOR It=0 TO 100
1090    J=It
1100    Th=X2-Dx*It                        !  THRESHOLD
1110    Mut=Mean_mucv+Th*Mean_sigcv
1120    Temp=Var_mucv+Th*Th*Var_sigcv
1130    Temp=Temp+2.*Th*SQR(Var_mucv*Var_sigcv)*Rho
1140    Sigmat=SQR(Temp)
1150    H1=C1+C2*(Mut-Ds)/Rs
1160    H2=C2*Sigmat/Rs
1170    Sa=FNS(A)
1180    Sb=FNS(B)
1190    PRINT "FNS(A) = ";Sa;"   FNS(B) = ";Sb
1200    DOUBLE N,K,J
1210    N=2
1220    H=(B-A)*.5
1230    S=(Sa+Sb)*.5
1240    V=1.E10
1250    T=0.
1260    FOR K=1 TO N-1 STEP 2
1270    T=T+FNS(A+H*K)
1280    NEXT K
1290    S=S+T
1300    Vo=V
1310    V=(S+T)*H*2./3.
1320    IF ABS(V-Vo)<1.E-9 THEN 1360
1330    N=N+N
1340    H=H*.5
1350    GOTO 1250
1360    Pd=F*V
1370    IF Pd>.9999995 THEN 1400
1380    P(It)=FNInvphi(Pd)
1390    NEXT It
1400    FOR It=0 TO J-1
1410    PLOT X2-Dx*It,P(It)
1420    NEXT It
1430    PENUP
1440    NEXT Ds
1450    PAUSE
1460    PRINTER IS CRT
1470    PLOTTER 505 IS TERMINATED
1480    END
1490    !
1500    DEF FNInvphi(X)         !  AMS 55, 26.2.23
1510    IF X=.5 THEN RETURN 0.
1520    P=MIN(X,1.-X)
1530    T=-LOG(P)
1540    T=SQR(T+T)
1550    P=1.+T*(1.432788+T*(.189269+T*.001308))
1560    P=T-(2.515517+T*(.802853+T*.010328))/P
1570    IF X<.5 THEN P=-P
1580    RETURN P
1590    FNEND
1600    !
1610    DEF FNS(X)
1620    COM H1,H2
1630    T=.5*X*X+EXP(H1+H2*X)
1640    IF T>100. THEN RETURN 0.
1650    RETURN EXP(-T)
1660    FNEND
```

Table E-3. Simulation - Extreme

```
10    Ns=16                          !  NUMBER OF SAMPLES
20    DATA 0,.75,1.5,2.25,3.,3.75,4.5,5.25,6.,6.75,7.5    !  DEFLECTION d
30    Rs=1.                          !  SCALING r
40    Nt=2^23                        !  NUMBER OF TRIALS
50    A$="E-16-75to75-1-23"          !  FILE NAME
60    Nb=1000                        !  NUMBER OF BINS
70    Mean=-.577215664902            !  FOR Vt, LOG OF EXPONENTIAL
80    Sigma=PI/SQR(6.)               !    RANDOM VARIABLE
90      Zmin=-15.                    !  MINIMUM Z VALUE
100     Zmax=5.                      !  MAXIMUM Z VALUE
110   DOUBLE Ns,Nt,Nb,Kt,Ks,J        !  INTEGERS
120   DIM P0(1000),P1(1000),P2(1000),P3(1000),P4(1000),P5(1000)
130   DIM P6(1000),P7(1000),P8(1000),P9(1000),P10(1000),Edf(1:11000)
140   READ D0,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10
150   A0=-Mean/Sigma
160   A1=1./Sigma
170   F=1./Ns
180   G=1./SQR(Ns-1)
190   Dz=(Zmax-Zmin)/Nb
200   FOR Kt=1 TO Nt                 !  SIMULATION
210   S1=S2=0.
220   FOR Ks=1 TO Ns
230   Vt=LOG(-LOG(RND))              !  UN-NORMALIZED RANDOM VARIABLE.
240   V=A1*Vt+A0                     !  ZERO MEAN, UNIT VARIANCE RV
250   S1=S1+V
260   S2=S2+V*V
270   NEXT Ks
280   Mc=S1*F                        !  SAMPLE MEAN
290   Sc=SQR(S2-Ns*Mc*Mc)*G          !  SAMPLE STANDARD DEVIATION
300   Vt=LOG(-LOG(RND))
310   V=A1*Vt+A0
320   C=Rs*V-Mc
330   Z=(D0+C)/Sc
340   J=INT((Z-Zmin)/Dz)
350   IF J<0 THEN J=0
360   IF J>Nb THEN J=Nb
370   P0(J)=P0(J)+1.
380   Z=(D1+C)/Sc
390   J=INT((Z-Zmin)/Dz)
400   IF J<0 THEN J=0
410   IF J>Nb THEN J=Nb
420   P1(J)=P1(J)+1.
430   Z=(D2+C)/Sc
440   J=INT((Z-Zmin)/Dz)
450   IF J<0 THEN J=0
460   IF J>Nb THEN J=Nb
470   P2(J)=P2(J)+1.
480   Z=(D3+C)/Sc
490   J=INT((Z-Zmin)/Dz)
500   IF J<0 THEN J=0
510   IF J>Nb THEN J=Nb
520   P3(J)=P3(J)+1.
```

```
530   Z=(D4+C)/Sc
540   J=INT((Z-Zmin)/Dz)
550   IF J<0 THEN J=0
560   IF J>Nb THEN J=Nb
570   P4(J)=P4(J)+1.
580   Z=(D5+C)/Sc
590   J=INT((Z-Zmin)/Dz)
600   IF J<0 THEN J=0
610   IF J>Nb THEN J=Nb
620   P5(J)=P5(J)+1.
630   Z=(D6+C)/Sc
640   J=INT((Z-Zmin)/Dz)
650   IF J<0 THEN J=0
660   IF J>Nb THEN J=Nb
670   P6(J)=P6(J)+1.
680   Z=(D7+C)/Sc
690   J=INT((Z-Zmin)/Dz)
700   IF J<0 THEN J=0
710   IF J>Nb THEN J=Nb
720   P7(J)=P7(J)+1.
730   Z=(D8+C)/Sc
740   J=INT((Z-Zmin)/Dz)
750   IF J<0 THEN J=0
760   IF J>Nb THEN J=Nb
770   P8(J)=P8(J)+1.
780   Z=(D9+C)/Sc
790   J=INT((Z-Zmin)/Dz)
800   IF J<0 THEN J=0
810   IF J>Nb THEN J=Nb
820   P9(J)=P9(J)+1.
830   Z=(D10+C)/Sc
840   J=INT((Z-Zmin)/Dz)
850   IF J<0 THEN J=0
860   IF J>Nb THEN J=Nb
870   P10(J)=P10(J)+1.
880   NEXT Kt
890   MAT P0=P0/(Nt)
900   MAT P1=P1/(Nt)
910   MAT P2=P2/(Nt)
920   MAT P3=P3/(Nt)
930   MAT P4=P4/(Nt)
940   MAT P5=P5/(Nt)
950   MAT P6=P6/(Nt)
960   MAT P7=P7/(Nt)
970   MAT P8=P8/(Nt)
980   MAT P9=P9/(Nt)
990   MAT P10=P10/(Nt)
1000  S0=S1=S2=S3=S4=S5=S6=S7=S8=S9=S10=0.
```

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```
1010   FOR J=Nb TO 0 STEP -1
1020   S0=S0+P0(J)
1030   IF S0=0. THEN 1060
1040   IF S0>=1. THEN 1070
1050   P0(J)=FNInvphi(S0)          !  P0(J)=Prob(Z0>=Zmin+J*Dz)
1060   NEXT J
1070   P0(J)=0.
1080   FOR J=Nb TO 0 STEP -1
1090   S1=S1+P1(J)
1100   IF S1=0. THEN 1130
1110   IF S1>=1. THEN 1140
1120   P1(J)=FNInvphi(S1)
1130   NEXT J
1140   P1(J)=0.
1150   FOR J=Nb TO 0 STEP -1
1160   S2=S2+P2(J)
1170   IF S2=0. THEN 1200
1180   IF S2>=1. THEN 1210
1190   P2(J)=FNInvphi(S2)
1200   NEXT J
1210   P2(J)=0.
1220   FOR J=Nb TO 0 STEP -1
1230   S3=S3+P3(J)
1240   IF S3=0. THEN 1270
1250   IF S3>=1. THEN 1280
1260   P3(J)=FNInvphi(S3)
1270   NEXT J
1280   P3(J)=0.
1290   FOR J=Nb TO 0 STEP -1
1300   S4=S4+P4(J)
1310   IF S4=0. THEN 1340
1320   IF S4>=1. THEN 1350
1330   P4(J)=FNInvphi(S4)
1340   NEXT J
1350   P4(J)=0.
1360   FOR J=Nb TO 0 STEP -1
1370   S5=S5+P5(J)
1380   IF S5=0. THEN 1410
1390   IF S5>=1. THEN 1420
1400   P5(J)=FNInvphi(S5)
1410   NEXT J
1420   P5(J)=0.
1430   FOR J=Nb TO 0 STEP -1
1440   S6=S6+P6(J)
1450   IF S6=0. THEN 1480
1460   IF S6>=1. THEN 1490
1470   P6(J)=FNInvphi(S6)
1480   NEXT J
1490   P6(J)=0.
1500   FOR J=Nb TO 0 STEP -1
1510   S7=S7+P7(J)
1520   IF S7=0. THEN 1550
1530   IF S7>=1. THEN 1560
1540   P7(J)=FNInvphi(S7)
1550   NEXT J
1560   P7(J)=0.
```

```
1570    FOR J=Nb TO 0 STEP -1
1580    S8=S8+P8(J)
1590    IF S8=0. THEN 1620
1600    IF S8>=1. THEN 1630
1610    P8(J)=FNInvphi(S8)
1620    NEXT J
1630    P8(J)=0.
1640    FOR J=Nb TO 0 STEP -1
1650    S9=S9+P9(J)
1660    IF S9=0. THEN 1690
1670    IF S9>=1. THEN 1700
1680    P9(J)=FNInvphi(S9)
1690    NEXT J
1700    P9(J)=0.
1710    FOR J=Nb TO 0 STEP -1
1720    S10=S10+P10(J)
1730    IF S10=0. THEN 1760
1740    IF S10>=1. THEN 1770
1750    P10(J)=FNInvphi(S10)
1760    NEXT J
1770    P10(J)=0.
1780    FOR J=1 TO Nb
1790    Edf(J)=P0(J)
1800    Edf(J+Nb)=P1(J)
1810    Edf(J+Nb*2)=P2(J)
1820    Edf(J+Nb*3)=P3(J)
1830    Edf(J+Nb*4)=P4(J)
1840    Edf(J+Nb*5)=P5(J)
1850    Edf(J+Nb*6)=P6(J)
1860    Edf(J+Nb*7)=P7(J)
1870    Edf(J+Nb*8)=P8(J)
1880    Edf(J+Nb*9)=P9(J)
1890    Edf(J+Nb*10)=P10(J)
1900    NEXT J
1910    CREATE DATA A$,396
1920    ASSIGN #1 TO A$
1930    PRINT #1;Edf(*)
1940    ASSIGN #1 TO *
1950    PAUSE
1960    END
1970    !
1980    DEF FNInvphi(X)              !   AMS 55, 26.2.23
1990    IF X=.5 THEN RETURN 0.
2000    P=MIN(X,1.-X)
2010    T=-LOG(P)
2020    T=SQR(T+T)
2030    P=1.+T*(1.432788+T*(.189269+T*.001308))
2040    P=T-(2.515517+T*(.802853+T*.010328))/P
2050    IF X<.5 THEN P=-P
2060    RETURN P
2070    FNEND
```

Table E-4. Plot - Simulation

```
10    A$="E-16-75to75-1-23"
20    Zmin=-15.
30    Zmax=5.
40    X1=-15.
50    X2=5.
60    Nb=1000
70    ASSIGN #1 TO A$
80    READ #1;Edf(*)
90    ASSIGN #1 TO *
100   DIM Edf(1:11000)
110   DIM A$[30],B$[30]
120   DIM Xlabel$(1:30),Ylabel$(1:30)
130   DIM Xcoord(1:30),Ycoord(1:30)
140   DIM Xgrid(1:30),Ygrid(1:30)
150   DOUBLE Nb,Lx,Ly,Nx,Ny,I,K        !  INTEGERS
160   !
170   A$="Threshold"
180   B$="Exceedance Probability"
190   !
200   Lx=11
210   REDIM Xlabel$(1:Lx),Xcoord(1:Lx)
220   DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
230   READ Xlabel$(*)
240   DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
250   READ Xcoord(*)
260   !
270   Ly=27
280   REDIM Ylabel$(1:Ly),Ycoord(1:Ly)
290   DATA E-6,E-5,E-4,.001,.002,.005,.01,.02,.05
300   DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
310   DATA .995,.998,.999,.9999,.99999,.999999
320   READ Ylabel$(*)
330   DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05
340   DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
350   DATA .995,.998,.999,.9999,.99999,.999999
360   READ Ycoord(*)
370   !
380   Nx=11
390   REDIM Xgrid(1:Nx)
400   DATA -15,-13,-11,-9,-7,-5,-3,-1,1,3,5
410   READ Xgrid(*)
420   !
430   Ny=27
440   REDIM Ygrid(1:Ny)
450   DATA 1.E-6,1.E-5,1.E-4,.001,.002,.005,.01,.02,.05
460   DATA .1,.2,.3,.4,.5,.6,.7,.8,.9,.95,.98,.99
470   DATA .995,.998,.999,.9999,.99999,.999999
480   READ Ygrid(*)
490   !
```

```
500     FOR I=1 TO Ly
510     Ycoord(I)=FNInvphi(Ycoord(I))
520     NEXT I
530     FOR I=1 TO Ny
540     Ygrid(I)=FNInvphi(Ygrid(I))
550     NEXT I
560     Y1=Ygrid(1)
570     Y2=Ygrid(Ny)
580     GINIT 180./240.                          !  VERTICAL PAPER
590     PLOTTER IS 505,"HPGL"
600     PRINTER IS 505
610     LIMIT PLOTTER 505,0.,180.,0.,240.        !  1 GDU = 2 mm
620     VIEWPORT 20.,120.,19.,132.
630   ! VIEWPORT 22.,85.,59.,122.                !  TOP OF PAPER
640   ! VIEWPORT 22.,85.,19.,62.                 !  BOTTOM OF PAPER
650     WINDOW X1,X2,Y1,Y2
660   !   PRINT "VS2"
670     FOR I=1 TO Nx
680     MOVE Xgrid(I),Y1
690     DRAW Xgrid(I),Y2
700     NEXT I
710     FOR I=1 TO Ny
720     MOVE X1,Ygrid(I)
730     DRAW X2,Ygrid(I)
740     NEXT I
750     PENUP
760     LDIR 0
770     CSIZE 2.3,.5
780     LORG 5
790     Y=Y1-(Y2-Y1)*.02
800     FOR I=1 TO Lx
810     MOVE Xcoord(I),Y
820     LABEL Xlabel$(I)
830     NEXT I
840     CSIZE 3.,.5
850     MOVE .5*(X1+X2),Y1-.06*(Y2-Y1)
860     LABEL A$
870     CSIZE 2.3,.5
880     LORG 8
890     X=X1-(X2-X1)*.01
900     FOR I=1 TO Ly
910     MOVE X,Ycoord(I)
920     LABEL Ylabel$(I)
930     NEXT I
940     LDIR PI/2.
950     CSIZE 3.,.5
960     LORG 5
970     MOVE X1-.15*(X2-X1),.5*(Y1+Y2)
980     LABEL B$
990       PRINT "VS36"
```

```
1000    Dz=(Zmax-Zmin)/Nb
1010    FOR I=1 TO Nb
1020    T=Edf(I)
1030    IF T=0. THEN 1050
1040    PLOT Zmin+Dz*I,T
1050    NEXT I
1060    PENUP
1070    FOR I=1 TO Nb
1080    T=Edf(I+Nb)
1090    IF T=0. THEN 1110
1100    PLOT Zmin+Dz*I,T
1110    NEXT I
1120    PENUP
1130    K=Nb*2
1140    FOR I=1 TO Nb
1150    T=Edf(I+K)
1160    IF T=0. THEN 1180
1170    PLOT Zmin+Dz*I,T
1180    NEXT I
1190    PENUP
1200    K=Nb*3
1210    FOR I=1 TO Nb
1220    T=Edf(I+K)
1230    IF T=0. THEN 1250
1240    PLOT Zmin+Dz*I,T
1250    NEXT I
1260    PENUP
1270    K=Nb*4
1280    FOR I=1 TO Nb
1290    T=Edf(I+K)
1300    IF T=0. THEN 1320
1310    PLOT Zmin+Dz*I,T
1320    NEXT I
1330    PENUP
1340    K=Nb*5
1350    FOR I=1 TO Nb
1360    T=Edf(I+K)
1370    IF T=0. THEN 1390
1380    PLOT Zmin+Dz*I,T
1390    NEXT I
1400    PENUP
1410    K=Nb*6
1420    FOR I=1 TO Nb
1430    T=Edf(I+K)
1440    IF T=0. THEN 1460
1450    PLOT Zmin+Dz*I,T
1460    NEXT I
1470    PENUP
```

```
1480    K=Nb*7
1490    FOR I=1 TO Nb
1500    T=Edf(I+K)
1510    IF T=0. THEN 1530
1520    PLOT Zmin+Dz*I,T
1530    NEXT I
1540    PENUP
1550    K=Nb*8
1560    FOR I=1 TO Nb
1570    T=Edf(I+K)
1580    IF T=0. THEN 1600
1590    PLOT Zmin+Dz*I,T
1600    NEXT I
1610    PENUP
1620    K=Nb*9
1630    FOR I=1 TO Nb
1640    T=Edf(I+K)
1650    IF T=0. THEN 1670
1660    PLOT Zmin+Dz*I,T
1670    NEXT I
1680    PENUP
1690    K=Nb*10
1700    FOR I=1 TO Nb
1710    T=Edf(I+K)
1720    IF T=0. THEN 1740
1730    PLOT Zmin+Dz*I,T
1740    NEXT I
1750    PENUP
1760    PAUSE
1770    PRINTER IS CRT
1780    PLOTTER 505 IS TERMINATED
1790    END
1800    !
1810    DEF FNInvphi(X)       !   AMS 55, 26.2.23
1820    IF X=.5 THEN RETURN 0.
1830    P=MIN(X,1.-X)
1840    T=-LOG(P)
1850    T=SQR(T+T)
1860    P=1.+T*(1.432788+T*(.189269+T*.001308))
1870    P=T-(2.515517+T*(.802853+T*.010328))/P
1880    IF X<.5 THEN P=-P
1890    RETURN P
1900    FNEND
```

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